CENTER OF MASS

CENTRE OF GRAVITY AND CENTRE OF MASS

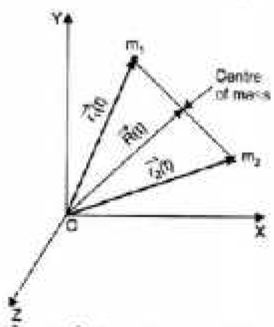
Centre of gravity of a body is defined as the point where the whole weight of the body can be supposed to act.

The centre of mass of a body is the point where the whole mass of the body can be supposed to be concentrated.

> *In a uniform gravitational field, the centre of mass and the centre of gravity coincide.*

THE CENTRE OF MASS OF A TWO-PARTICLE SYSTEM

Consider a system consisting of two point masses (particles) m_1 and m_2 whose position vectors at a time t with reference to the origin O of the inertial frame are $\vec{r_1}(t)$ and $\vec{r_2}(t)$ respectively



The total force $(\vec{F_1})_{total}$ acting on the point mass m_1 comprises of two parts :

(i) A force $(\vec{F_1})_{ext.}$ which is due to some agency external to the system.

(ii) A force $\vec{F_{12}}$ due to point mass m_2. This force is an internal force of the system.

$$\therefore \quad (\vec{F_1})_{total} = \vec{F_{12}} + (\vec{F_1})_{ext.} \qquad ...(1)$$

Similarly, for the point mass m_2

$$(\vec{F_2})_{total} = \vec{F_{21}} + (\vec{F_2})_{ext.} \qquad ...(2)$$

Now, the equation of motion of point mass m_1, on the basis of Newton's second law of motion, is

$$\frac{d}{dt}(m_1\vec{v_1}) = (\vec{F_1})_{total} \qquad ...(3)$$

Similarly, for point mass m_2

$$\frac{d}{dt}(m_2\vec{v_2}) = (\vec{F_2})_{total} \qquad ...(4)$$

Adding (3) and (4), we get

$$\frac{d}{dt}(m_1\vec{v_1}) + \frac{d}{dt}(m_2\vec{v_2}) = (\vec{F_1})_{total} + (\vec{F_2})_{total}$$

$$\frac{d}{dt}(m_1\vec{v_1} + m_2\vec{v_2}) = (\vec{F_1})_{total} + (\vec{F_2})_{total}$$

From (1) and (2),

$$\frac{d}{dt}(m_1\vec{v_1} + m_2\vec{v_2}) = \vec{F_{12}} + (\vec{F_1})_{ext.} + \vec{F_{21}} + (\vec{F_2})_{ext.}$$

But $\vec{F_{21}} = -\vec{F_{12}}$ (Newton's third law of motion)

$$\therefore \quad \frac{d}{dt}(m_1\vec{v_1} + m_2\vec{v_2}) = (\vec{F_1})_{ext.} + (\vec{F_2})_{ext.}$$

or

$$\frac{d}{dt}(m_1\vec{v_1} + m_2\vec{v_2}) = \vec{F} \quad \text{...(5)}$$

where $\vec{F}$ is the total external force. It is given by

$$\vec{F} = (\vec{F_1})_{ext.} + (\vec{F_2})_{ext.}$$

It may be pointed out here that $(\vec{F_1})_{ext.}$ and $(\vec{F_2})_{ext.}$ act at different points of the system. But they are being added as free vectors.

The velocity vectors are given by $\vec{v_1} = \frac{d\vec{r_1}}{dt}$ and $\vec{v_2} = \frac{d\vec{r_2}}{dt}$.

$$\text{Now, } m_1\vec{v_1} + m_2\vec{v_2} = m_1\frac{d\vec{r_1}}{dt} + m_2\frac{d\vec{r_2}}{dt}$$

$$= \frac{d}{dt}(m_1\vec{r_1}) + \frac{d}{dt}(m_2\vec{r_2})$$

$$= \frac{d}{dt}(m_1\vec{r_1} + m_2\vec{r_2})$$

From equation (5), $\vec{F} = \frac{d^2}{dt^2}(m_1\vec{r_1} + m_2\vec{r_2})$

or

$$\vec{F} = M\frac{d^2}{dt^2}\left(\frac{m_1\vec{r_1} + m_2\vec{r_2}}{M}\right)$$

This equation is clearly the equation of motion of a hypothetical object of mass $M (= m_1 + m_2)$. Its position at any time is given by position vector $\vec{R}(t)$ such that

$$\vec{R}(t) = \frac{m_1\vec{r_1}(t) + m_2\vec{r_2}(t)}{m_1 + m_2} \quad \text{...(6)}$$

The point whose position is defined by $\vec{R}(t)$ is called the centre of mass of the two-particle system.

CO-ORDINATES OF THE CENTRE OF MASS

We know that $\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \ldots\ldots + m_n\vec{r_n}}{M}$

Let (x, y, z) be the co-ordinates of the centre of mass of the system. Let (x_1, y_1, z_1), (x_2, y_2, z_2), ..., (x_n, y_n, z_n) be the co-ordinates of the n-particle system.

$$\text{Then } X\hat{i} + Y\hat{j} + Z\hat{k} = \frac{1}{M}[m_1x_1\hat{i} + m_1y_1\hat{j} + m_1z_1\hat{k} + m_2x_2\hat{i} + m_2y_2\hat{j} + m_2z_2\hat{k} + \ldots\ldots + m_nx_n\hat{i} + m_ny_n\hat{j} + m_nz_n\hat{k}]$$

$$= \frac{1}{M}[(m_1x_1 + m_2x_2 + \ldots\ldots + m_nx_n)\hat{i} + (m_1y_1 + m_2y_2 + \ldots\ldots + m_ny_n)\hat{j} + (m_1z_1 + m_2z_2 + \ldots\ldots + m_nz_n)\hat{k}]$$

Comparing coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$X = \frac{m_1x_1 + m_2x_2 + \ldots\ldots + m_nx_n}{M} = \frac{\sum_{i=1}^{n} m_ix_i}{M},$$

$$Y = \frac{m_1y_1 + m_2y_2 + \ldots\ldots + m_ny_n}{M} = \frac{\sum_{i=1}^{n} m_iy_i}{M}$$

$$Z = \frac{m_1z_1 + m_2z_2 + \ldots\ldots + m_nz_n}{M} = \frac{\sum_{i=1}^{n} m_iz_i}{M}$$

MOMENTUM CONSERVATION AND MOTION OF CENTRE OF MASS

The equation of motion of centre of mass is : $\vec{F} = \frac{d}{dt}(M\vec{V})$

where $\vec{F}$ is the net external force acting on the system, M is the mass of the system and $\vec{V}$ is the velocity of the centre of mass of the system.

If no external force acts on the system, then $\vec{F} = 0$.

$$\therefore \quad \frac{d}{dt}(M\vec{V}) = 0$$

| $\because$ Differential coefficient of a constant quantity is zero.

$$\Rightarrow \quad M\vec{V} = \text{constant} \quad \ldots(1)$$

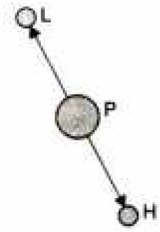
$$\text{or} \quad m_1\vec{v_1} + m_2\vec{v_2} + \ldots\ldots + m_n\vec{v_n} = \text{constant} \quad \ldots(2)$$

It is clear from equations (1) and (2) that **if no external force acts on the system, then the total momentum of the system is constant.** This is law of conservation of momentum.

It is further clear from equation (1) that $\vec{V}$ = constant ($\because$ M is constant.)

EXAMPLES OF MOTION OF CENTRE OF MASS

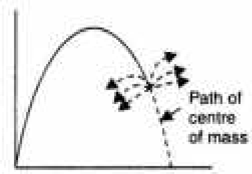
(1) Consider the spontaneous decay of a radioactive nucleus P into two fragments. The parent nucleus P was initially at rest (Fig.). The decay is caused by the internal forces of the system. So, the centre of mass of the decay products continues to be at rest. Since no external force acts on the system, therefore, momentum is conserved. The centre of mass has zero momentum both before and after the explosion.



Spontaneous decay of a radioactive nucleus

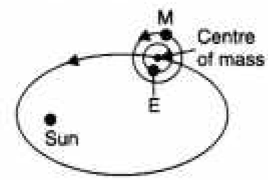
After the explosion, the lighter and the heavier fragments L and H respectively move in opposite directions. The lighter fragment has a proportionately larger speed than the heavier mass.

(2) Consider the case of a projectile which explodes in flight. The force of gravity continues to act on the total mass of the projectile (*now scattered into fragments*). The centre of mass of the fragments continues along the parabolic trajectory. It is the same as was followed by the intact projectile. This idealised illustration will be altered somewhat by the presence of air resistance. *The air resistance is larger for irregularly-shaped fragments than for the projectile.*



Motion of centre of mass of exploding projectile

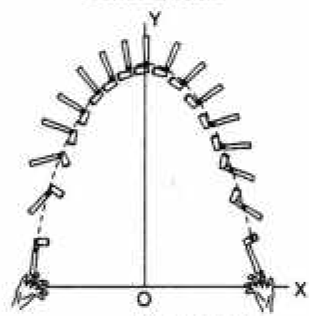
In this illustration, the forces of explosion are all internal forces. These forces are exerted by part of the system on other parts of the system. These forces may change the momenta of all the individual fragments from the values they had when they made up the projectile. But the internal forces cannot change the total vector momentum of the system. It is only the external force which can change the total momentum of the system. In the given problem, the only external force is that due to gravity . The change in the total momentum of the system due to gravity is the same whether the shell explodes or not.



Centre of mass of Earth-Moon system

(3) Consider the Earth-Moon system. Both the Earth and the Moon move in circles about their centre of mass, always being on opposite sides of it. The centre of mass moves along an elliptical path around the Sun. The forces of attraction between Earth and Moon are internal to the Earth-Moon system. On the other hand, the Sun's attraction of both Earth and Moon are external forces.

(4) An axe is tossed between two performers and rotates as it travels. The parabolic path of the centre of mass (represented by the dot on the axe) is indicated by the dashed line. A particle tossed in the same way would follow that same path. No other point on the axe moves in such a simple way



Tossing of axe

(5) A missile containing three warheads follows a parabolic path. An explosion releases the three warheads, which travel so that their centre of mass follows the original parabolic path

Exploding missile

1. Two particles of masses 3 kg and 1 kg are located at $(-6\hat{i} + 4\hat{j} - 2\hat{k})$ and $(2\hat{i} + 5\hat{j} + 13\hat{k})$ respectively. Locate the position of centre of mass.

Solution. $M\vec{R} = m_1\vec{r_1} + m_2\vec{r_2}$

or $4\vec{R} = 3(-6\hat{i} + 4\hat{j} - 2\hat{k}) + 1(2\hat{i} + 5\hat{j} + 13\hat{k})$

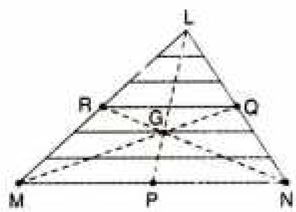
or $4\vec{R} = -16\hat{i} + 17\hat{j} + 7\hat{k}$

or $\vec{R} = -4\hat{i} + \frac{17}{4}\hat{j} + \frac{7}{4}\hat{k}$ ◀

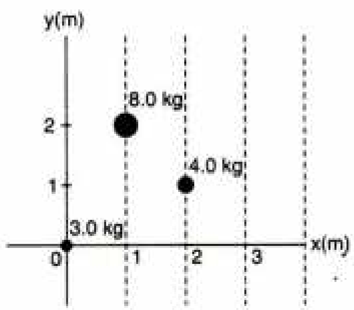
2. Find the centre of mass of a triangular lamina.

Solution. The lamina (ΔLMN) may be subdivided into narrow strips each parallel to the base (MN) as shown in Fig. 15.16.

By symmetry, each strip has its centre of mass at its midpoint. If we join the midpoints of all the strips, we get the median LP. The centre of mass of the triangle as a whole, therefore, has to lie on the median LP. Similarly, we can argue that it lies on the median MQ and NR. This means the centre of mass lies on the point of concurrence of the medians, *i.e.*, on the **centroid** G of the triangle. ◀



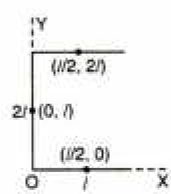
3. What are the co-ordinates of the centre of mass of the three-particle system shown in



Solution. $x = \frac{3 \times 0 + 4 \times 2 + 8 \times 1}{3 + 4 + 8}$ m $= \frac{16}{15}$ m $=$ **1.1 m**

$y = \frac{3 \times 0 + 4 \times 1 + 8 \times 2}{3 + 4 + 8}$ m $= \frac{20}{15}$ m $= \frac{4}{3}$ m $=$ **1.3 m** ◀

4. Given : A U-shaped uniform wire of sides $2l$, l and l. The x and y co-ordinates of the centre of mass of each side are shown in Fig. Calculate the x and y co-ordinates of the centre of mass of wire.

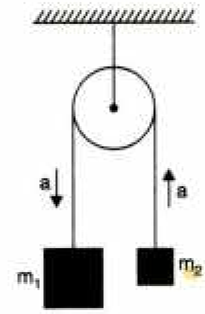


Solution. $X = \frac{m \times \frac{l}{2} + 2m \times 0 + m \times \frac{l}{2}}{m + 2m + m} = \frac{ml}{4m} = \frac{l}{4}$

$Y = \frac{m \times 0 + 2m \times l + m \times 2l}{m + 2m + m} = \frac{4ml}{4m} = l$

So, the x and y co-ordinates of the centre of mass are $\left(\frac{l}{4}, l\right)$. ◀

5. Two bodies of masses m_1 and m_2 ($< m_1$) are connected to the ends of a massless cord and allowed to move as shown. The pulley is both massless and frictionless. Determine the acceleration of the centre of mass.



Solution. If $\vec{a}$ is the acceleration of m_1, then $-\vec{a}$ is the acceleration of m_2.

We know that

$(m_1 + m_2)\vec{A}_{cm} = m_1\vec{a_1} + m_2\vec{a_2}$

or $(m_1 + m_2)\vec{A}_{cm} = m_1\vec{a} - m_2\vec{a}$

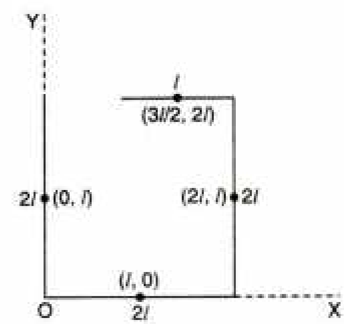
or $\vec{A}_{cm} = \frac{m_1 - m_2}{m_1 + m_2}\vec{a}$

But $\vec{a} = \frac{m_1 - m_2}{m_1 + m_2}\vec{g}$

[connected-body problem]

∴ $\vec{A}_{cm} = \frac{(m_1 - m_2)^2}{(m_1 + m_2)^2}\vec{g}$ ◀

6. A wire of uniform cross-section is bent in the shape shown in Fig. 15.20. The co-ordinates of the centre of mass of each side are shown in Fig. 15.20. What are the co-ordinates of the centre of mass of the system ?



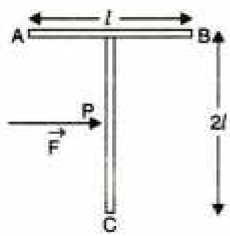
Solution. $X = \dfrac{2m \times l + 2m \times 2l + m \times \frac{3l}{2} + 2m \times 0}{2m + 2m + m + 2m}$

$$= \frac{2l + 4l + \frac{3l}{2}}{7} = \frac{15l}{14}$$

$$Y = \frac{2m \times 0 + 2m \times l + m \times 2l + 2m \times l}{7m} = \frac{6ml}{7m} = \frac{6l}{7}$$

So, the co-ordinates of the centre of mass are $\left(\frac{15l}{14}, \frac{6l}{7}\right)$. ◀

7. A 'T' shaped object with dimensions shown in the Fig. 15.21 is lying on a smooth floor. A force $\vec{F}$ is applied at the point P parallel to AB, such that the object has only the translational motion without rotation. Find the location of P with respect to C. [AIEEE 2005]



Solution. Since the object has only the translational motion without rotation therefore the centre of mass of the object is the point where the force has been applied. So, we have to find the centre of mass of the object.

Let us take C as the origin and CD to be along Y-axis. If m be the mass of AB, then the mass of CD is $2m$. The centre of mass of AB is at a distance $2l$ from C. The centre of mass of CD is at a distance l from C.

Distance of centre of mass of the object from C

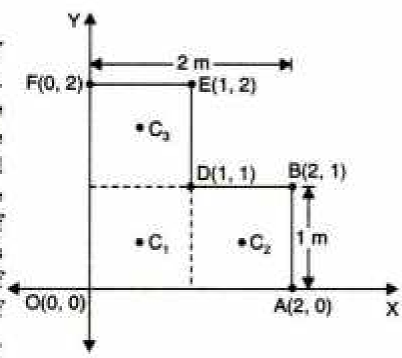
$$= \frac{2m \times l \times m \times 2l}{2m + m} = \frac{4ml}{3m} = \frac{4l}{3}$$ ◀

Note. *This question has also been included in the next chapter with a different solution.*

8. Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown. The mass of the lamina is 3 kg.

Solution.

Choosing the X and Y axes as shown in Fig. 15.22 we have the coordinates of the vertices of the L-shaped lamina as given in the figure. We can think of the L-shape to consists of 3 squares each of length 1 m. The mass of each square is 1 kg, since the lamina is uniform. The centres of mass C_1, C_2 and C_3 of the squares are, by symmetry, their geometric centres and have coordinates (1/2, 1/2), (3/2, 1/2), (1/2, 3/2) respectively. We take the masses of the squares to be concentrated at these points. The centre of mass of the whole L shape (X, Y) is the centre of mass of these mass points.



$$X = \frac{[1(1/2) + 1(3/2) + 1(1/2)]\text{ kg m}}{(1 + 1 + 1)\text{ kg}} = \frac{5}{6}\text{ m}$$

$$Y = \frac{[1(1/2) + 1(1/2) + 1(3/2)]\text{ kg m}}{(1 + 1 + 1)\text{ kg}} = \frac{5}{6}\text{ m}$$

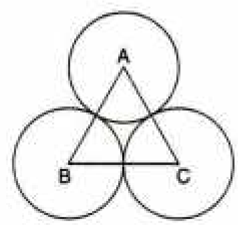
The centre of mass of the L-shape lies on the line OD. ◀

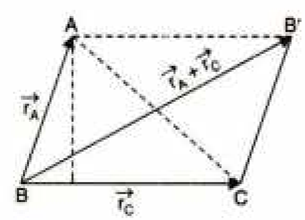
We could have guessed this without calculations. Can you tell why ? Suppose, the three squares that make up the L shaped lamina had different masses. How will you then determine the centre of mass of the lamina ?

9. Three identical spheres, each of radius R, are placed touching each other on a horizontal table. Where is the centre of mass of the system located ? Assume that mass distribution is uniform in each sphere.

Solution. The centre of mass of each sphere is at its geometrical centre. So, the centre of mass of the system is in fact the centre of mass of three equal point masses located at the vertices of an equilateral triangle ABC. Here A, B and C are the centres of three spheres. Each side of the triangle is 2R. In order to locate the centre of mass of the system, we have to choose one point, say B, as the origin. The position vector of the centre of mass with respect to the origin B is given by

$$\vec{R}_{cm} = \frac{M\vec{r}_A + M\vec{r}_C}{3M}$$, where M is the mass of each sphere.





or $\vec{R}_{cm} = \dfrac{\vec{r}_A + \vec{r}_C}{3}$

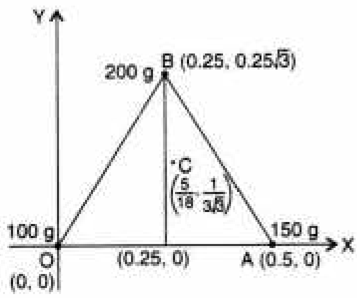
$\vec{BB'} = \vec{r}_A + \vec{r}_C$

(Parallelogram law of vectors)

The centre of mass is located at the point of intersection of the medians of the triangle. ◀

10. Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100 g, 150 g and 200 g respectively. Each side of the equilateral triangle is 0.5 m long.

Solution. With the X and Y axes chosen as shown in Fig. 15.25, the coordinates of points O, A and B forming the equilateral triangle are respectively (0, 0), (0.5, 0), $(0.25, 0.25\sqrt{3})$.



Let the masses 100 g, 150 g and 200 g be located at O, A and B be respectively. Then,

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{[100(0) + 150(0.5) + 200(0.25)]\ \text{gm}}{(100 + 150 + 200)\ \text{g}}$$

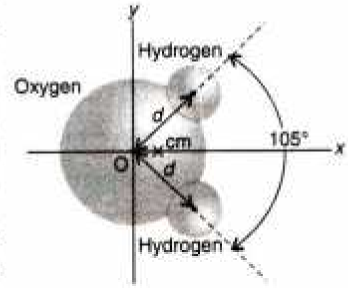
$$= \frac{75 + 50}{450}\ \text{m} = \frac{125}{450}\ \text{m} = \mathbf{\frac{5}{18}\ m}$$

$$Y = \frac{[100(0) + 150(0) + 200(0.25\sqrt{3})]\ \text{gm}}{450\ \text{g}}$$

$$= \frac{50\sqrt{3}}{450}\ \text{m} = \frac{\sqrt{3}}{9}\ \text{m} = \mathbf{\frac{1}{3\sqrt{3}}\ m}$$

The centre of mass C is shown in the figure. Note that it is not the geometric centre of the triangle OAB. ◀

11. Fig. 15.26 shows a simple model of the structure of a water molecule. The separation between atoms is $d = 9.57 \times 10^{-11}$ m. Each hydrogen atom has mass 1.0 u and the oxygen atom has mass 16.0 u. Find the position of the centre of mass.



Solution. Nearly the whole of the mass of each atom is concentrated in its nucleus, which is only about 10^{-5} times the overall radius of the atom. So, we can safely represent each atom as a point.

The x-coordinate of each hydrogen atom is $d \cos (105°/2)$; the y-coordinates of the upper and lower hydrogen atoms are $+ d \sin (105°/2)$ and $- d \sin (105°/2)$, respectively. The coordinates of the oxygen atom are $x = 0, y = 0$.

$$x_{cm} = \frac{\left[\begin{array}{l}(1.0\,u)(d \cos 52.5°) + (1.0\,u) \\ \times (d \cos 52.5°) + (16.0\,u)(0)\end{array}\right]}{1.0\,u + 1.0\,u + 16.0\,u}$$

$$= 0.068\,d = 0.068 \times 9.57 \times 10^{-11}\ \text{m}$$

$$= \mathbf{6.5 \times 10^{-12}\ m}$$

$$y_{cm} = \frac{\left[\begin{array}{l}(1.0\,u)(d \sin 52.5°) + (1.0\,u) \\ \times (-d \sin 52.5°) + (16.0\,u)(0)\end{array}\right]}{1.0\,u + 1.0\,u + 16.0\,u}$$

$$= \mathbf{0.}$$ ◀

The centre of mass is much closer to the oxygen atom than to either hydrogen atom because the oxygen atom is much more massive. Notice that the centre of mass lies along the x-axis, which is the ***axis of symmetry*** *of this molecule. If the molecule is rotated by 180° around this axis, it looks exactly the same as before. The position of the centre of mass cannot be affected by this rotation, so it must lie on the axis of symmetry.*