Oscillations

PERIODIC, OSCILLATIONS AND VIBRATIONS. A motion that repeats itself at regular intervals of time is called periodic motion. If there is an equilibrium position somewhere within the path of periodic motion, the motion is called oscillatory vibrating motion.
Conventionally, when the frequency is small, it is called oscillatory motion. If the frequency is high, it is called vibratory motion.
Circulatory motion is periodic but not oscillatory.

SIMPLE HARMONIC MOTION & CHARACTERISTICS

(1) Simple harmonic motion is a special type of periodic motion, in which a particle move to and fro repeatedly about a means potion

(2) in linear S.H.M a restoring force act on the particle which is always directed towards the means potion at the instant i.e. restoring force displacement of the particles from means potion

$$F \propto -x \Rightarrow F = -kx$$

Where k is known as force constant or spring constant or stiffness. Its S.I unit is newton/meter and dimensional formula $[MT^{-2}]$.

(3) Instead of straight like motion, if particle of centre of mass of body is oscillating on a small arc of circular path, then for angular S.H.M
Restoring torque (τ) ∝ - angular displacement (θ)
Negative sign indicates direction of force (or torque) is opposite to displacement.

(4) Time period is independent of amplitude in SHM.

DIFFERENTIAL EQUATION OF SIMPLE HARMONIC MOTION.

In SHM, F = -kx $\Rightarrow m\frac{d^2x}{dt^2} = -kx$

Or, $\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0 \Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$

where $\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{k/m}$

The above equation is called differential equation of SHM.
For angular S.H.M $\tau = -c\theta$ and

$\boxed{\frac{d^2\theta}{dt^2} + \omega^2\theta = 0}$ Where $\omega^2 = \frac{c}{I}$ [As c = Restoring torque constant and I = Moment of inertia]

BASIC TERMS RELATED TO SHM.

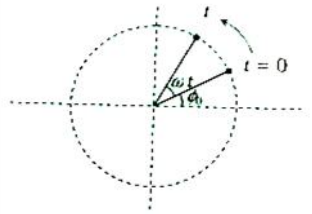
(1) Period (T) And Frequency (f or ν). The smallest interval of time after which the motion is repeated is called the period. The number of oscillations/ vibration / repetition of motion is called frequency.

Thus, $f = \frac{1}{T}$

(2) Angular Frequency(ω). Angular frequency of a body executing periodic motion is equal to product of frequency of the body with factor 2π. Angular frequency ω = 2πf. Its unit is rad/sec.

(3) Amplitude. The maximum displacement of oscillating particle from its equilibrium position is called amplitude.

(4) Phase(∅).
In the equation $y = a \sin(\omega t + \emptyset_0)$
Here $(\omega t + \emptyset_0)$ is called phase of vibrating particle. & $\emptyset_0$ = Initial phase or epoch. It is the phase of vibrating particle at t = 0.



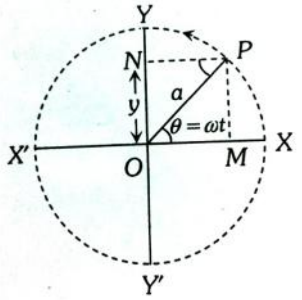
(a) Same Phase. Two vibrating particles are said to be in same phase, if the phase difference between them is an even multiple of π (e.g., 2 π, 4π, 6 π….) or path difference is an even multiple of (λ/2) (e.g., λ, 2λ, 3λ….) or time interval is an even multiple of (T/2).

(b) Opposite phase. Two vibrating particles are said to be in opposite phase, if the phase difference between them is an odd multiple of π (e.g., π, 3π, 5 π….) or path difference is an odd multiple of (λ/2) (e.g., λ/2, 3λ/2, 5λ/2….) or time interval is an odd multiple of (T/2).

(c) Phase difference. If two particles perform S.H.M and their equations are
$y_1 = a\sin(\omega t + \emptyset_1)$ and $y_2 = a\sin(\omega t + \emptyset_2)$
then phase difference $\Delta\emptyset = (\omega t + \emptyset_2) - (\omega t + \emptyset_1)$
$= \emptyset_2 - \emptyset_1$

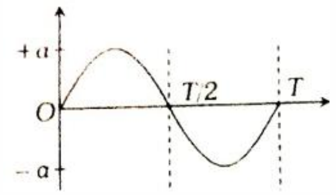
DISPLACEMENT IN S.H.M.

The displacement of a particle executing S.H.M at an instant is defined as the distance of particle from the mean position at that instant.

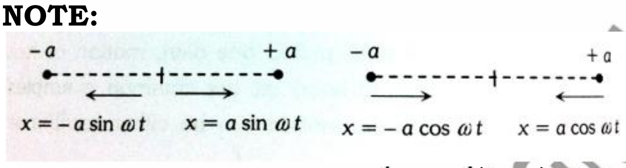


From figure, sinωt= $\frac{O}{O} = \frac{y}{a}$

$$\boxed{y = a\sin\omega t = a\sin\frac{2}{T}t = a\sin 2\pi nt}$$

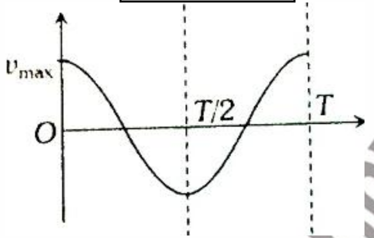


NOTE:



VELOCITY IN S.H.M. $v = \frac{d}{d} = a\frac{d(s\ \ \omega t)}{d}$

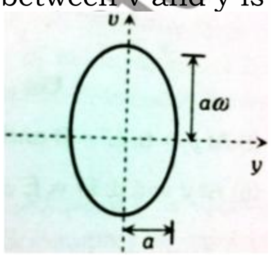
$$\boxed{v = a\omega\cos\omega t}$$



or, $\boxed{v = a\omega\sqrt{1-\sin^2\omega t} = \omega\sqrt{a^2-y^2}}$

$\Rightarrow \frac{v^2}{\omega^2} = a^2 - y^2 \Rightarrow \frac{v^2}{a^2\omega^2} + \frac{y^2}{a^2} = 1$

This is the equation of ellipse. Hence graph between v and y is an **ellipse**. This is also true for momentum- position graph. The momentum position graph is known as **phase-space** graph. If **ω = 1**, graph between v and y is a **circle**.



Sp.Case.(a) At mean position y = 0 & ωt =0

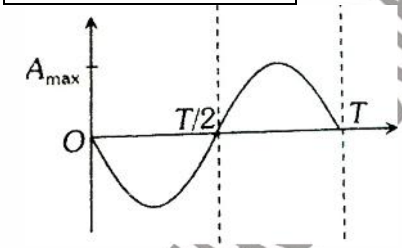
$v_{max} = a\omega$

(b) At extreme position y = ±a & ωt = π/2

$v_m = 0$

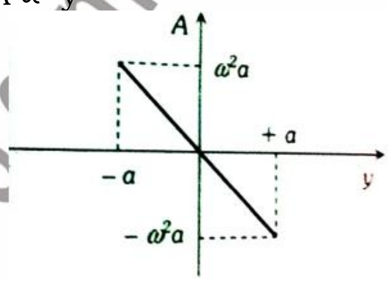
ACCELERATION IN S.H.M

$$\boxed{A = \frac{d}{d} = \frac{d}{d}(a\omega\cos\omega t)}$$



$$\boxed{A = -\omega^2 a\sin\omega t = -\omega^2 y}$$

i.e., A ∝ -y



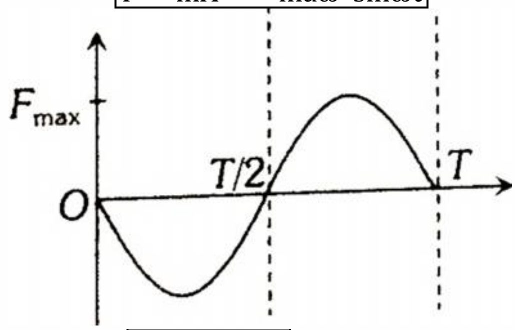
Graph between acceleration (A) and displacement (y) is a straight line as shown, slope of the line = $-\omega^2$

Sp. Case.(a) At mean position y = 0,
i.e., at t = 0 or t = T/2, ωt=π, $\boxed{A_m = 0}$

(b) At extreme position y = ± a,
i.e., t = T/4 or ωt = $\frac{\pi}{2}$, $\boxed{|A_m| = \omega^2 y}$

FORCE IN S.H.M .

$$\boxed{F = mA = -ma\omega^2\sin\omega t}$$



$\boxed{F = -m\omega^2 y}$ where $\omega = \sqrt{k/m}$

TIME PERIOD IN S.H.M

(a) From equation of acceleration

$$|A| = \omega^2 y \Rightarrow \omega = \sqrt{\frac{A}{Di}}$$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{Di}{Ac}}}$$

(b) From equation $\omega = \sqrt{k/m}$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{\text{I}_1 \quad \text{f}_i \quad (m)}{\text{S} \quad \text{f}_i \quad (k)}}}$$

KINETIC ENERGY IN S.H.M.
This is because of the velocity of the particle .

Kinetic energy (K) $= \frac{1}{2}mv^2$

$$\boxed{K = \frac{1}{2}m\omega^2(a^2 - y^2)}$$

Also $\boxed{K = \frac{1}{2}m\omega^2\cos^2\omega t}$

Sp.case. (a) At mean position y = 0,
t = 0, ωt = 0 $\boxed{K_m = \frac{1}{2}m\omega^2a^2}$

(b) At extreme position y = ± a
t = T/4, ωt = π/2 $\boxed{K_m = 0}$

POTENTIAL ENERGY IN S.H.M.
The work done by the conservative force on the particle is stored up as P.E of the particle.

$F = -\frac{d}{d}$, Where U = Potential Energy

$\therefore dU = - F\,dy = -(-m\omega^2y)dy = m\omega^2y\,dy$

$U = \int_0^y m\omega^2y\,dy = m\omega^2\left[\frac{y^2}{2}\right]_0^y = \frac{1}{2}m\omega^2y$

$$\boxed{U = \frac{1}{2}m\omega^2y}$$

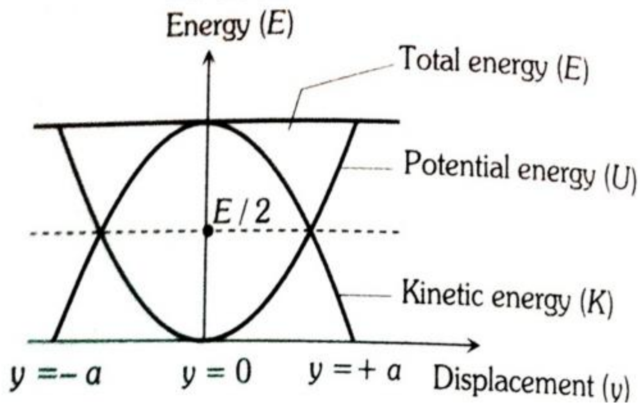
Also $\boxed{U = \frac{1}{2}m\omega^2a^2\sin^2\omega t}$

Sp.case. (a) At mean position y = 0,
t = 0, ωt = 0 $\boxed{U_m = 0}$

(b) At extreme position y = ± a
t = T/4, ωt = π/2 $\boxed{U_m = \frac{1}{2}m\omega^2a}$

TOTAL ENERGY IN S.H.M.
T.E = K.E + P.E $= \frac{1}{2}m\omega^2a$
i.e total energy not a function of position.

ENERGY POSITION GRAPH.



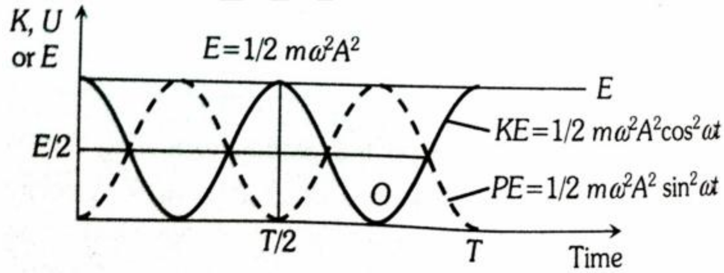
(i) At $y = 0$; $U = 0$ and $K = E$

(ii) At $y = \pm a$; $U = E$ and $K = 0$

(iii) At $y = \pm\frac{a}{2}$; $U = \frac{E}{4}$ and $K = \frac{3E}{4}$

(iv) At $y = \pm\frac{a}{\sqrt{2}}$; $U = K = \frac{E}{2}$

ENERGY TIME GRAPH.



AVERAGE VALUE OF P.E AND K.E
The average value of potential energy for complete cycle is given by

$$U_a = \frac{1}{T}\int_0^T U\,dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2a^2\sin^2\omega t\,dt = \frac{1}{4}m\,\omega^2a^2$$

The average value of kinetic energy for complete cycle is given by

$$K_a = \frac{1}{T}\int_0^T K\,dt$$

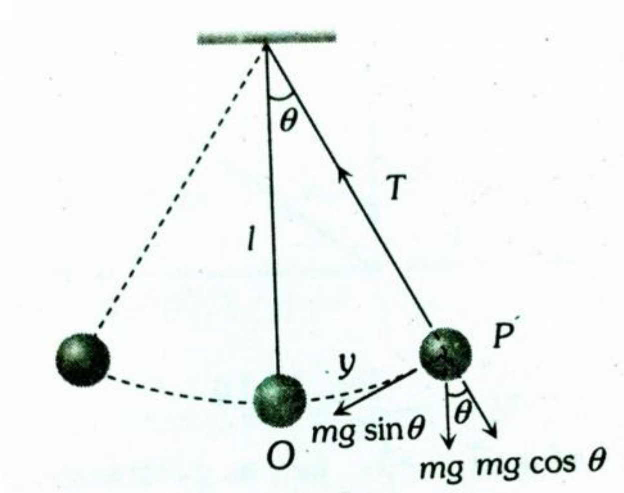
$$= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2a^2\cos^2\omega t\,dt = \frac{1}{4}m\,\omega^2a^2$$

Thus average value of K.E and P.E of harmonic oscillator are equal and each equal to the half of the total energy

$$K_a = U_a = \frac{1}{2}E$$

SIMPLE PENDULUM
A heavy point mass suspended from a rigid support using inextensible, elastic and mass less thread and free to oscillate without friction is called a simple pendulum.
All these conditions are ideal and cannot be realised completely in practice. Hence such pendulum is also called mathematical pendulum.

Motion of a simple pendulum



Restoring force, F = -mg sinθ = -mg θ (if θ is small and measured in radians)

$$F = -mg.\frac{y}{l} = -\frac{m}{l}.y\text{.........}(1)$$

∴ F ∝ -y , Hence motion is S.H.M

$$\text{But } F = -m\omega^2 y \quad \text{...............}(2)$$

Comparing equation (1) & (2) we get,

$\Rightarrow \omega = \sqrt{\frac{g}{l}}$ $\quad \Rightarrow$ $\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$

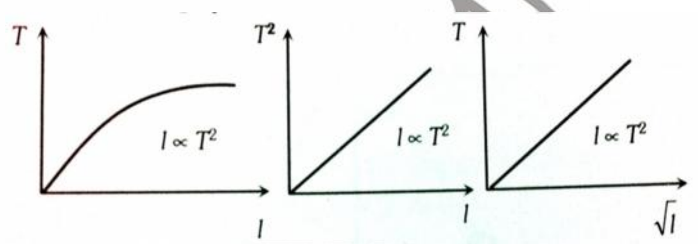
FACTOR AFFECTING TIME PERIOD OF SIMPLE PENDULUM

(a) Mass of the bob: Time period of simple pendulum is independent of mass of the bob.

If the solid bob is replaced by a hollow sphere of same radius but different mass, time period remains unchanged.

If a girl is swinging in a swing and another sits with her, the time period remains unchanged.

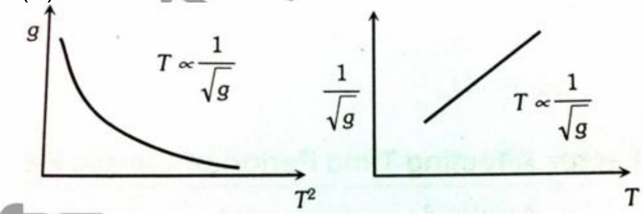
(b) Length of the pendulum: Time period $T \propto \sqrt{l}$ where I is the distance between point of suspension and center of mass of bob and is called effective length.

- When a sitting girl on a swinging swing stands up, her center of mass will go up and so l and hence T will decrease.
- If a hole is made at the bottom of a hollow sphere full of water and water comes out slowly through the hole and time period is recorded till the sphere is empty, initially and finally the center of mass will be at the center of the sphere. However, as water drains off the sphere, the center of mass of the system will move down. Due to this l and hence T increase, reaches a maximum and then again becomes equal to its initial value when sphere is empty.
- 

(c) Effect of g:

$T \propto \frac{1}{\sqrt{g}}$ i.e., g increase T decrease.

- As we go high above the earth's surface or we go deep inside the mines the value of g decreases, hence time period of pendulum (T) increases.



(d) Effect of temperature on time period: If the bob of simple pendulum is suspended by a wire then effective length of pendulum will increase with the rise of temperature due to which the time period will increase.

$l = l_0(1 + \alpha\Delta\theta)$

Where Δθ = rise in temperature

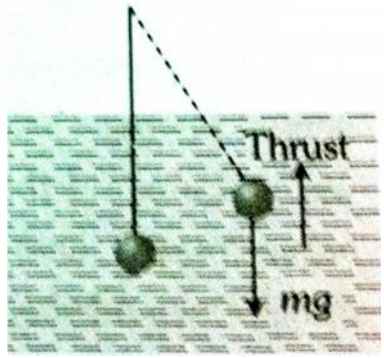
l_0 = initial length, l = final length of the wire.

$$\frac{T}{T_0} = \sqrt{\frac{l}{l_0}} = (1 + \alpha\Delta\theta)^{1/2} = 1 + \frac{1}{2}\alpha\Delta\theta$$

So $\frac{T}{T_0} - 1 = \frac{1}{2}\alpha\Delta\theta$ i.e., $\boxed{\frac{\Delta T}{T} \approx \frac{1}{2}\alpha\Delta\theta}$

OSCILLATION OF PENDULUM IN DIFFERENT SITUATION.

(1) Oscillation in liquid: If bob of a simple pendulum of density ρ is made to oscillate in some fluid of density σ (ρ > σ) then time period of simple pendulum gets increased.



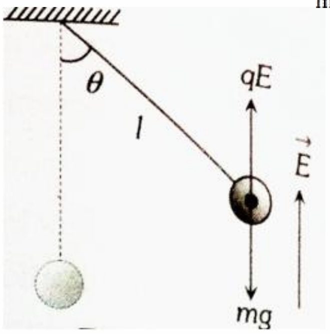
$V\rho g_e = V\rho g - V\sigma g \quad \Rightarrow \frac{g_e}{g} = \frac{\rho-\sigma}{\rho}$

$\Rightarrow \frac{T'}{T} = \sqrt{\frac{g}{g_e}} = \sqrt{\frac{\rho-\sigma}{\rho}} > 1$

(2) Oscillation under the influence of electric field: If a bob of mass m carries a positive charge q and pendulum is placed in uniform electric field of strength E.

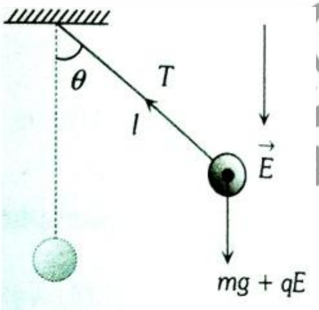
(i) If electric field is directed vertically upwards.

Effective acceleration $g_e = g - \frac{q}{m}$



So $\boxed{T = 2\pi\sqrt{\frac{l}{g-\frac{q}{m}}}}$ So, T will increase.

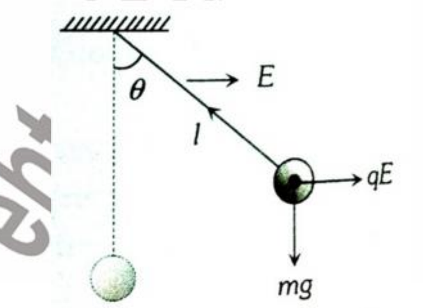
(ii) If electric field is directed vertically downward.



$g_e = g + \frac{q}{m}$,

$\boxed{T = 2\pi\sqrt{\frac{l}{g+\frac{q}{m}}}}$, T will decrease.

(iii) If electric field in horizontal direction.



$g_e = \sqrt{g^2 + \left(\frac{q}{m}\right)^2}$,

$\boxed{T = \sqrt{\frac{l}{\left[g^2+\left[\frac{qE}{m}\right]^2\right]^{1/2}}}}$, T will decrease.

(3) Pendulum in a lift: If pendulum is suspended from the ceiling of the lift.

(i) If the lift is at rest or moving upward with constant velocity.

$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$, Time period unchanged.

(ii) If the lift is moving upward with constant acceleration a.

$\boxed{T = 2\pi\sqrt{\frac{l}{g+a}}}$, Time period decreases.

(iii) If the lift is moving downward with constant acceleration a.

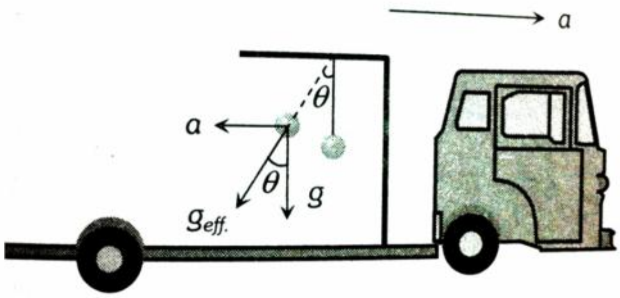
$\boxed{T = 2\pi\sqrt{\frac{l}{g-a}}}$, Time period increases.

(iv) If the lift is moving downward with acceleration a = g.

$\boxed{T = 2\pi\sqrt{\frac{l}{g-g}} = \infty}$

(4) Pendulum in an accelerated vehicle.

The time period of simple pendulum whose point of suspension moving horizontally with acceleration a.



$g_e = \sqrt{g^2 + a^2} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{l}{(g^2+a^2)^{1/2}}}}$

And $\boxed{\theta = \tan^{-1}\left(\frac{a}{g}\right)}$

If simple pendulum suspended in a car that is moving with constant speed v around a circle of radius r.

$\boxed{T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+\left(\frac{v^2}{r}\right)^2}}}}$

SOME OTHER TYPES OF PENDULUM

(1) Infinite length pendulum.

If length of the pendulum is comparable to the radius of earth then

$$T = 2\pi\sqrt{\frac{1}{g\left[\frac{1}{l}+\frac{1}{R}\right]}}$$

(i) If $l << R$, then $\frac{1}{l} >> \frac{1}{R}$ so $T = 2\pi\sqrt{\frac{l}{g}}$

(ii) If $l >> R (\rightarrow \infty)$ then $\frac{1}{l} << \frac{1}{R}$

so $T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6.4\times10^6}{10}} \cong 84.6$ minutes

and it is the maximum time period which an oscill simple pendulum can have

(iii) If $l = R$ so $T = 2\pi\sqrt{\frac{R}{2g}} \cong 1$hour

(2) Second pendulum.

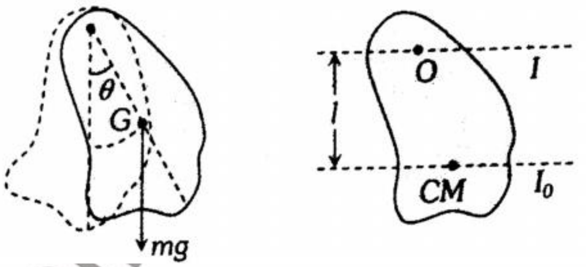
It is that simple pendulum whose time period of vibration is two seconds.
Putting T = 2s and g = 9.8 m/s² in

$T = 2\pi\sqrt{\frac{l}{g}}$

we get $l = \frac{4\times9.8}{4\pi^2} = 99.3\ cm$ Or nearly 1 meter.

For moon the length of second's pendulum will be 1/6 meter [As $g_m = \frac{g_e}{6}$]

(3) Compound pendulum: Any rigid body suspended from a fixed support constitute a physical pendulum. Consider the situation when the body is displaced through a small angle θ. Torque on the body about O is given by $\tau = mgl\sin\theta$......(i)
Where l = distance between point of suspension and centre of mass of the body.



If I be the MOI the body about O, $\tau = I\alpha$ ....(ii)
Where α = Angular accaleration.

From (i) and (ii), we get $I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$

as θ and $\frac{d^2\theta}{dt^2}$ are oposite to each other.

$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta$, since θ is very small .

Comparing with equation $\frac{d^2\theta}{dt^2} = -\omega^2\theta$, we get

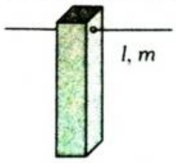
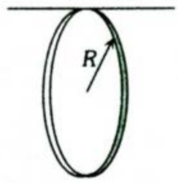
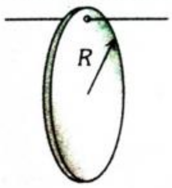
$\omega = \sqrt{\frac{mgl}{I}} \Rightarrow T = 2\pi\sqrt{\frac{I}{mgl}}$ Also $I = I_c + ml^2$

$= mk^2 + ml^2$ (where k = radius of gyration)

$\therefore T = 2\pi\sqrt{\frac{mk^2+ml^2}{mgl}} = 2\pi\sqrt{\frac{\frac{k^2}{l}+l}{g}} = 2\pi\sqrt{\frac{l_{eff}}{g}}$

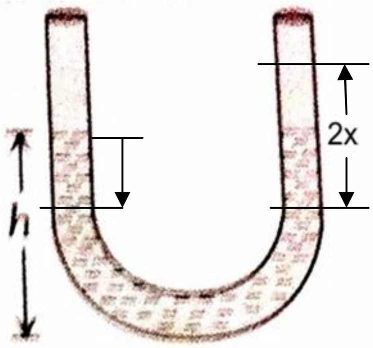
l_{eff} = Effective length of pendulum = distance between point of suspension and centre of mass.

SOME COMMON PHYSICAL PENDULUM

	Body	Time period
Bar		$T = 2\pi\sqrt{\frac{2l}{3g}}$
Ring		$T = 2\pi\sqrt{\frac{2R}{g}}$
Disc		$T = 2\pi\sqrt{\frac{3R}{2g}}$

Various formulae of S.H.M

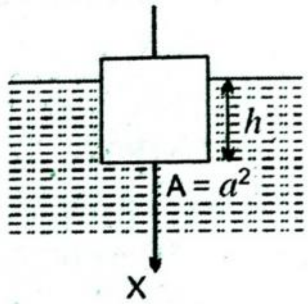
(1) S.H.M of a liquid in U tube.



Mass of liquid in U tube = 2hAρ
Mass of oscillating liquid = 2xAρ
Restoring force F = - (2xAρ)g .....(1)
i.e., F ∝ -x hence motion is S.H.M
But F = - Kx ....(2)

$\therefore K = 2A\rho g$, $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2hA\rho}{2A\rho g}} = 2\pi\sqrt{\frac{h}{g}}$

(2) S.H.M of a floating cubical block :
Let h be the immersed depth of cube initially. Ah = displaced volume



σAhg = weight of displaced volume = buoyant force

$a^3\rho g$ = weight of block

Equilibrium of the cube means

$$a^3\rho g = \sigma a^2 hg$$

$$a\rho = h\sigma \qquad \ldots(1)$$

In depressed state by x,

$$F_x = a^3\rho g - (h+x)a^2\sigma g \qquad \ldots(2)$$

From (1) and (2)

$$F_x = -(a^2\sigma g)x$$

This shows that the motion is SHM. Comparing with standard equation $F_x = -m\omega^2 x$

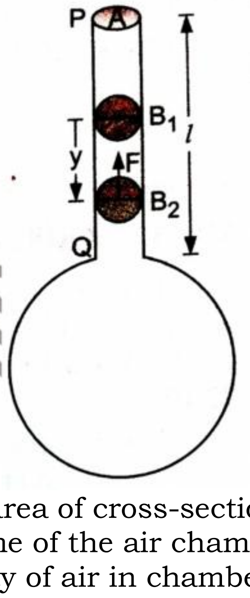
we have

$$-a^3\rho\omega^2 x = -a^2\sigma g x$$

$$\omega^2 = \frac{\sigma g}{\rho a}$$

$$\therefore \quad \omega = \sqrt{\frac{\sigma g}{\rho a}}$$

(3) Oscillation of ball in air chamber.
Let PQ represent the neck of an air chamber in which a ball is resting at B_1.



Let , A = area of cross-section of neck,
V = Volume of the air chamber,
P = density of air in chamber,
B = bulk modulus of elasticity of air,
m = mass of the ball.
Consider the ball be displaced from B_1 to B_2 where , B_1B_2 = y. As a result of this, air inside the chamber is compressed. If ΔV is the decreased in volume,
ΔV = Volume of air column of length B_1B_2 = Ay

We know that bulk modulus of elasticity,

$$B = \frac{\text{stress}}{\text{volumetric strain}} \quad \text{or} \quad B = \frac{F/A}{-\Delta V/V}$$

(negative sign here denotes decrease in volume)

$$\text{or} \quad F = -\frac{BA\Delta V}{V} = -\frac{BA(Ay)}{V} = -\left(\frac{BA^2}{V}\right)y \qquad \ldots(1)$$

If d^2y/dt^2 is the acceleration produced in the ball, then

$$F = m(d^2y/dt^2) \qquad \ldots(2)$$

From eqns. (1) and (2),

$$m\frac{d^2y}{dt^2} = -\left(\frac{BA^2}{V}\right)y$$

$$\text{or} \quad \frac{d^2y}{dt^2} = -\left(\frac{BA^2}{mV}\right)y = -\omega^2 y$$

$$\text{or} \quad \frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \ldots(3)$$

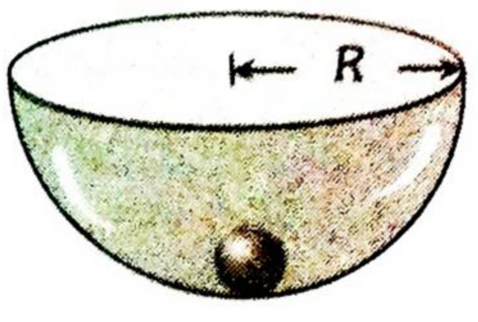
$$\text{where} \quad \omega^2 = \frac{BA^2}{mV}$$

$$\text{or} \quad \omega = \sqrt{\frac{BA^2}{mV}}$$

Eqn. (3) represents an SHM. Thus, the ball executes SHM and its time period is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{BA^2/mV}} = 2\pi\sqrt{\frac{mV}{BA^2}} \qquad \ldots(4)$$

(4) S.H.M of a small ball rolling down in hemispherical bowl.



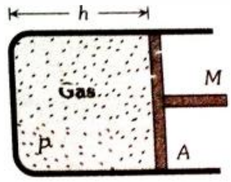
$$T = 2\pi\sqrt{\frac{R-r}{g}}$$

Where R= radius of the bowl
r = radius of the ball

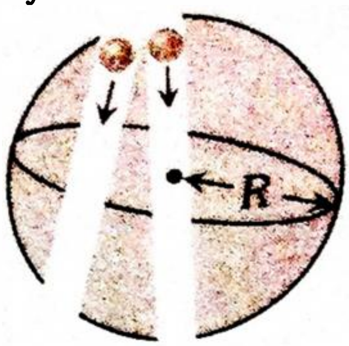
(5) S.H.M of piston in a cylinder.

$T = 2\pi\sqrt{\frac{Mh}{PA}}$

M = mass of the piston
A = area of cross section
h = height of cylinder
P = pressure in a cylinder

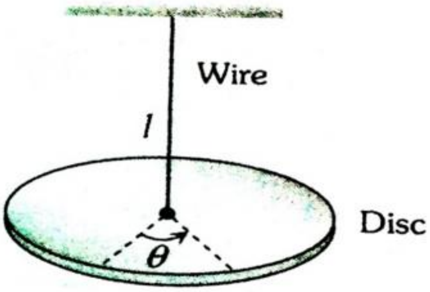


(6) S.H.M of a body in a tunnel dug along any chord of earth.



$$T = 2\pi\sqrt{\frac{R}{g}} = 84.6 \text{ minute}$$

(7) Tortional pendulum .



In a torsional pendulum an object is suspended from a wire. If such a wire is twisted, due to elasticity it exerts a restoring torque τ = Cθ.

$T = 2\pi\sqrt{\frac{I}{C}}$ Where I = Moment Inertia

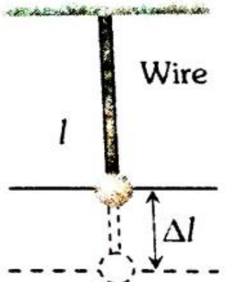
$C = T \quad c \quad = \frac{\eta\pi r^4}{2l}$

η = Modulus of elasticity, r = radius of wire
l = length of wire.

(8) Longitudinal oscillations of an elastic wire.

Wire/String pulled a distance Δl and left. It executes longitudinal oscillations.

Restoring force $F = -A \left(\frac{\Delta l}{l}\right)$, Y = Young's modulus, A = Area of cross-section



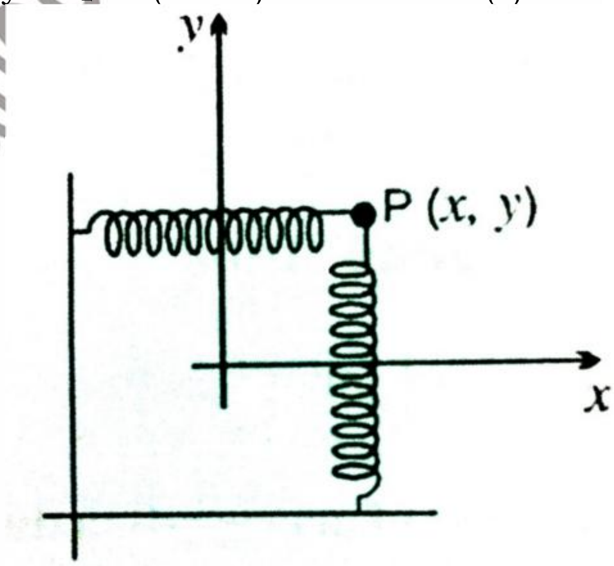
Hence $T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{m}{A}}$

SUPERPOSITION OF PERPENDICULAR S.H.M. (LISSAJOUS FIGURES)

Let us cosider two SHM's acting on the same particle along perpendicular direction. The resulting motion of the particle will be obtained by eliminating t from the expression for x and y.

Let $x = A_1 \sin\omega t$ ....(i)

$y = A_2 \sin(\omega t + \alpha)$ ....(ii)



From (ii) $\frac{y}{A_2} = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha$

$= \frac{x}{A_1}\cos\alpha + \sqrt{1 - \frac{x^2}{A_1^2}}\sin\alpha$ [using (i)]

$\Rightarrow \left(\frac{y}{A_1} - \frac{x}{x_1}\cos\alpha\right)^2 = \left(1 - \frac{x^2}{A_1^2}\right)\sin^2\alpha$

$\Rightarrow \frac{y^2}{A_2^2} + \frac{x^2}{A_1^2}\cos^2\alpha - \frac{2}{A_1A_2}\cos\alpha$

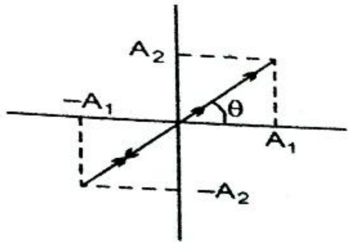
$= \sin^2\alpha - \frac{x^2}{A_1^2}\sin^2\alpha$

$\Rightarrow \frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} - \frac{2}{A_1A_2}\cos\alpha = \sin^2\alpha$

This is an equation of ellipse.

Special case :

Case (i) α = 0 In this case we have

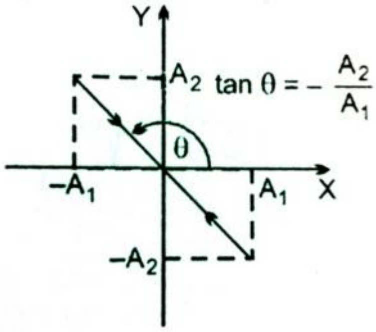


$\frac{y^2}{A_2^2}+\frac{x^2}{A_1^2}-\frac{2}{A_1A_2}=0 \Rightarrow \left(\frac{y}{A_2}-\frac{x}{A_1}\right)^2=0 \Rightarrow y=\frac{A_2}{A_1}x$

The path is a straight line segment (see figure) $\tan\theta=\frac{A_2}{A_1}$

This motion in SHM with amplitude $\sqrt{A_1^2+A_2^2}$

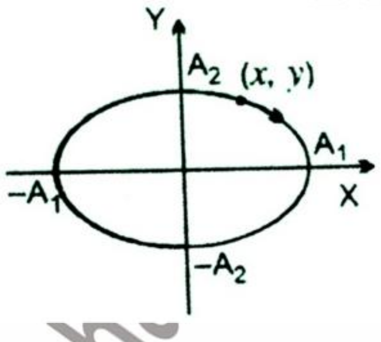
Case (ii) $\alpha=\pi$, **In this case, we have**



$\frac{y^2}{A_2^2}+\frac{x^2}{A_1^2}+\frac{2}{A_1A_2}=0 \;\Rightarrow y=-\frac{A_2}{A_1}x$

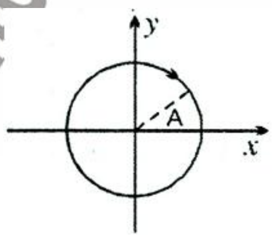
The path is a straight line segment inclined at and θ where the amplitude is $\sqrt{A_1^2+A_2^2}$

Case (iii) $\alpha=\pi/2$ In this case, we have



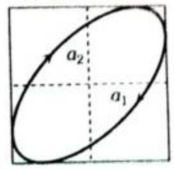
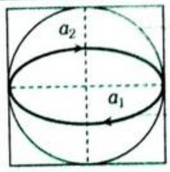
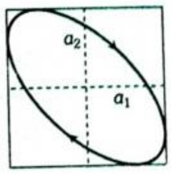
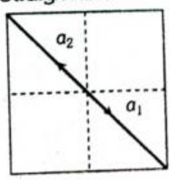
$\frac{x^2}{A_2^2}+\frac{x^2}{A_1^2}=1$, This is an ellipse.

Case (iv) $\alpha=\pi/2$, $A_1=A_2$ In this case we have,



$x^2+y^2=A^2$, This is a cicle.

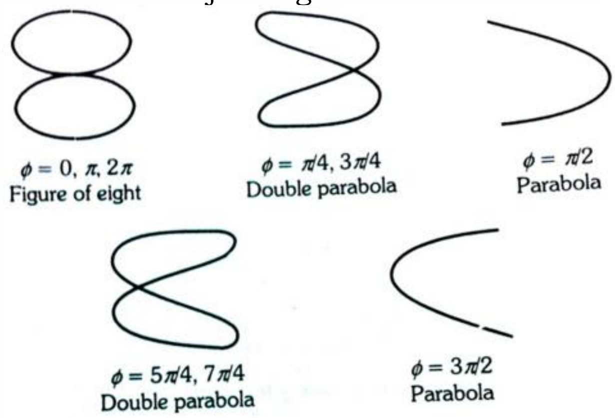
LISSAJOUS FIGURE IN OTHER CONDITIONS $\left(\frac{\omega_1}{\omega_2}=1\right)$

Phase diff.(ϕ)	Equation	Figure
$\frac{\pi}{4}$	$\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}-\frac{\sqrt{2}xy}{a_1a_2}=\frac{1}{2}$	Oblique ellipse 
$\frac{\pi}{2}$	$\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}=1$	 $a_1=a_2$ (Circle) $a_1\neq a_2$ (Ellipse)
$\frac{3\pi}{4}$	$\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}-\frac{\sqrt{2}xy}{a_1a_2}=\frac{1}{2}$	Oblique ellipse 
π	$\frac{x}{a_1}+\frac{y}{a_2}=0$ $\Rightarrow y=-\frac{a_2}{a_1}x$	Straight line 

For the frequency ratio ω1 : ω2 = 2 : 1 the two perpendicular SHM's are

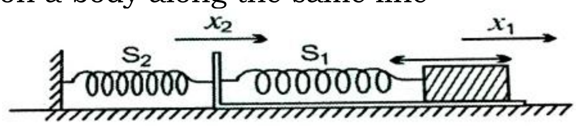
$x=a_1\sin(\omega_1 t+\emptyset)\,a \quad y=a_2\sin\omega_2$

Different Lissajous figure are as follows

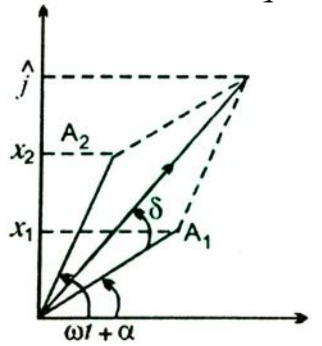


SUPERPOSITION OF SHM IN SAME DIRECTION

Let two SHMs of equal frequency be acting on a body along the same line



$x_1 = A_1\sin\omega t$ & $x_2 = A_2\sin(\omega t + \alpha)$
$= A_1\sin\omega t + A_2\sin(\omega t + \alpha)$
using the rotation vector $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$, we have

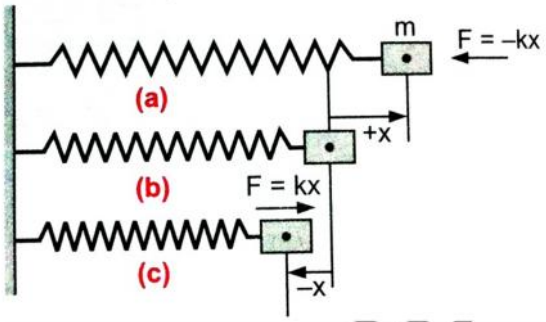


$\therefore x_1 = \overrightarrow{A_1}.\hat{j}$ and $x_2 = \overrightarrow{A_2}.\hat{j}$, $x = \overrightarrow{A_1}.\hat{j} + \overrightarrow{A_2}.\hat{j}$
$= (\overrightarrow{A_1} + \overrightarrow{A_2})\hat{j} = A\sin(\omega t + \delta)$
$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\alpha$
Where $\tan\delta = \frac{A_2\sin\alpha}{A_1 + A_2\cos\alpha}$

SPRING MASS SYSTEM

(i) Horizontal position.

When mass is displaced through a small distance x, a restoring force F = -kx ...(1)
When the mass is released it start oscillating back and forth about the equilibrium position under the influence of this restoring force.

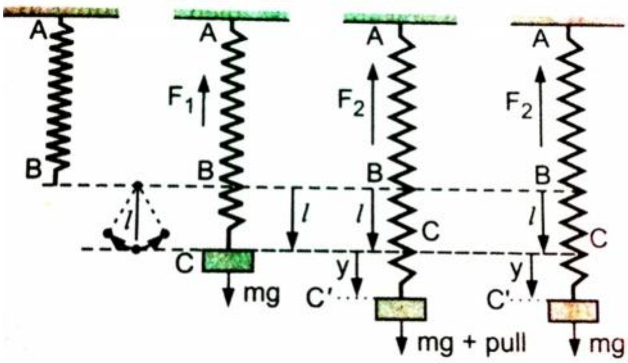


But $F = -m\omega^2 x$ ....(2)

Equating (1) and (2) we get $\omega = \sqrt{\frac{k}{m}}$

Or $T = 2\pi\sqrt{\frac{m}{k}}$

(ii) Vertical position.



If F_1 be the restoring force set up in the spring, then $F_1 = -kl$
{ As the system is in equilibrium
$F_1 + mg = 0 \therefore k = \frac{m}{l}$ }
If F_2 is the restoring force set up in the spring, when it is pulled through a small distance y, then $F_2 = -k(l + y)$
So, effective restoring force
$F = F_1 + F_2 = -k(l + y) - kl = -ky$ ....(1)
But $F = -m\omega^2 x$ ....(2)

Equating (1) and (2) we get $\omega = \sqrt{\frac{k}{m}}$

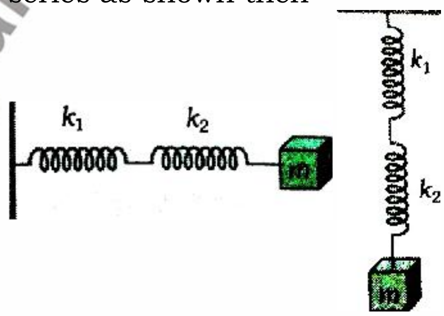
Or $\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$

Puttin the value of k we get, $\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$

The time period of a loaded massless spring is equal to that of a simple pendulum whose length is equal to the extension in the spring.

COMBINATION OF SPRING.

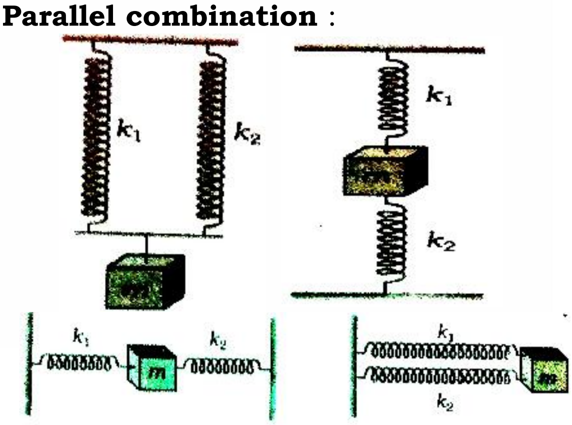
(i) Series combination. If two springs of spring constants K_1 and K_2 are joined in series as shown then



(a) In series combination equal forces acts on spring but extension in springs are different.

(b) $\boxed{\frac{1}{K_s} = \frac{1}{K_1} + \frac{1}{K_2}}$

(ii) Parallel combination :



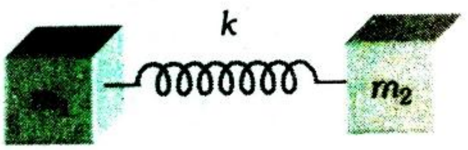
(a) In parallel combination different forces acts on different spring but extension in springs are same.

(b) $K_P = K_1 + K_2$

MASSIVE SPRING :

If the spring has a mass M and mass m is suspended from it, effective mass is given by $m_e = m + \frac{M}{3}$. Hence $T = 2\pi\sqrt{\frac{m_e}{K}}$

REDUCED MASS : If two masses of mass m1 and m2 are connected by a spring and made to oscillate on horizontal surface, the reduced mass m_r is given by $\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}$ so that



$$T = 2\pi\sqrt{\frac{m_r}{K}}$$

NOTE: If a spring of length l is cut into n pieces of length $l_1, l_2, l_3, \ldots\ldots l_n$, then these pieces have spring constants equal to $\frac{1}{l_1}K, \frac{1}{l_2}K, \frac{1}{l_3}K, \ldots\ldots\ldots\ldots, \frac{1}{l_n}K$ respectively.

FREE OSCILLATION.

(i) The oscillation of a particle with fundamental frequency under the influence of restoring force are are defined as free oscillations.

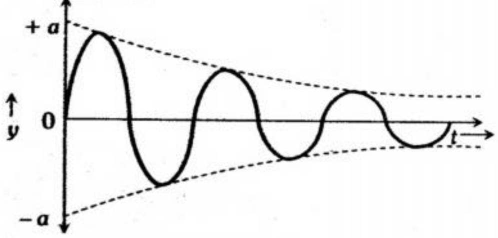
(ii) The amplitude, frequency and energy of oscillation remains constant.

DAMPED OSCILLATION.

(i) The oscillation of a body whose amplitude goes on decreasing with time is defined as damped oscillation.

(ii) In these oscillation the amplitude of oscillation decreases exponentially due to damping forces like frictional force, viscous force, hystrissis etc.

(iii) Due to decrease in amplitude the energy of the oscillator also goes on decreasing exponentially.



(iv) The force pruducing a resistance to the oscillation is called damping force.

If the velocity of oscillator is v then Damping force

$F_d = -bv$, b = damping constant.

(v) Resultant force on damping oscillator is given by $F = F_R + F_d = -K\ \ - b$

$\square$ $m\frac{d^2x}{d\ ^2} + b\frac{d}{d} + K\ \ = 0$

(vi) Displacement of damped oscillator is given by $x = x_m e^{-(\frac{b}{2})t} s\ \ (\omega' t + \square)$

Where ω' = angular frequency of the damped oscillator = $\sqrt{\omega_0^2 - (b/2\ \ )^2}$

The amplitude decreases continuously with time according to $x = x_m e^{-(\frac{b}{2})t}$

(vii) For a damped oscillator if the damping is small then the mechaincal energy decreases exponentially with time as $E = \frac{1}{2}Kx_m^2 e^{-b\ /m}$.

FORCED OSCILLATION.

(i) The oscillation in which a body oscillates under the influence of an external periodic force is known as forced oscillation.

(ii) The amplitude of oscillator would have decreeases due to damping forces but on account of the energy gained from the external source it remains constant.

(iii) In forced oscillation, frequency of damped oscillator is equal to the frequency of external force.

(iv) Suppose an external driving force is represented by $F(t) = F_0\cos\omega_d t$ The motion of a particle under combined action of

(a) Restoring force (-Kx)

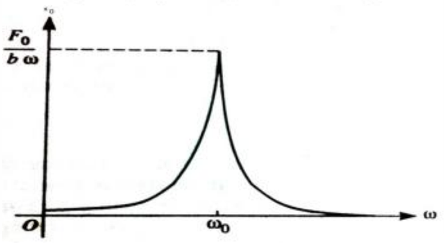
(b) Damped force (-bv) and

(c) Driving force F(t) is given by

$ma = -kx - bv + F_0\cos\omega_d t$

The solution of this equation gives

$x = x_0\sin(\omega_d t + \square)$ with amplitude



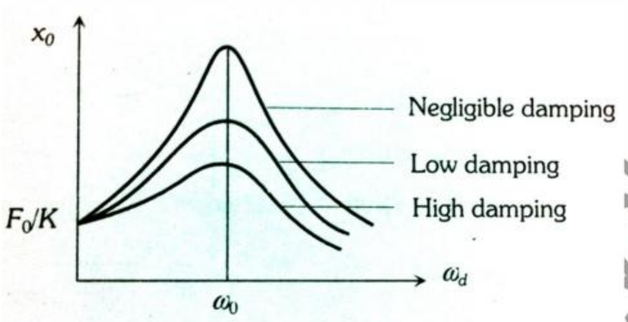
$$x_0 = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_d^2) + (\frac{b\omega}{m})^2}}$$ and

$$t\iota = \frac{(\omega_0^2-\omega_d^2)}{b\omega_d/m}$$

where $\omega_0 = \sqrt{\frac{K}{m}}$ = Natural frequency.

RESONANCE. When the frequency of external force is equal to the natural frequency of the oscillator. Then this state is known as the state of resonance. And this frequency is known as resonant frequency.

AMPLITUDE RESONANCE . The amplitude of forced oscillator depend upon the frequency ω_d of external force. When $n\omega_d = \omega_0$, the amplitude is maximum but not infinite because of presence of damping force. The corresponding frequency is called resonant frequency (ω_0)



MAINTAINED OSCILLATION.

The oscillation in whics the loss of energy of oscillator is compensated by the supplying energy from an external source are known as maintained oscillation.

Example :

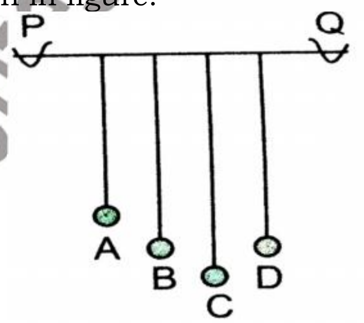
(i) the balance wheel of a watch where the main spring supplies the external energy.

(ii) Electrically maintained tuning fork

(iii) An electronic oscillator.

ILLUSTRATION OF FREE, FORCED AND RESONMANT OSCILLATIONS.

Suspended four simple pendulums A, B, C and D with light bobs from a rubber string stretched between two fixed point P and Q as shown in figure.



The length of the pendulum B and D are equal. The pendulum A is shorter, whereas the pendulum C is longer than the pendulums B and D.

Displaced the bob of the pendulum D to one side and release it so as to set it into vibrations. These vibrations are communicated to all other pendulums through rubber string and as a result, all of them start vibrating. The vibrations of A and C are irregular to start with but after some time these settle to vibrate with the frequency of D. The amplitude of vibrations of B goes on increasing till it becomes equal to that of D. Clearly, the vibrations of D are free vibrations. Since the pendulums b and D have the same length, their natural time periods and frequencies are the same. The vibrations of B are, therefore, resonant vibrations. The pendulum A and C, Which have diffrent lengths, are made to vibrate with the frequency of D, which is different from their natural frequencies. Hence, the vibrations of A and C are forced vibrations.