PROBLEM BASED ON NEWTON'S LAW OF GRAVITATION

1. Calculate the force of attraction between two balls each of mass 1 kg each, when their centres are 10 cm apart. Given $G = 6.67 \times 10^{-11}\ Nm^2kg^{-2}$. **[Delhi 1997]**

Solution. Here $m_1 = m_2 = 1$ kg, $r = 10$ cm $= 0.10$ m

$$\therefore \quad F = \frac{G\, m_1 m_2}{r^2} = \frac{6.67 \times 10^{-11} \times 1 \times 1}{(0.10)^2}$$

2. The mass of planet Jupiter is 1.9×10^{27} kg and that of the sun is 1.99×10^{30} kg. The mean distance of the Jupiter from the Sun is 7.8×10^{11} m. Calculate the gravitational force which the sun exerts on Jupiter. Assuming that Jupiter moves in a circular orbit around the sun, calculate the speed of the Jupiter.

Solution. Here $M = 1.99 \times 10^{30}$ kg,

$m = 1.9 \times 10^{27}$ kg, $r = 7.8 \times 10^{11}$ m,

$G = 6.67 \times 10^{-11}\ \text{Nm}^2\ \text{kg}^{-2}$

$$\therefore \quad F = \frac{GMm}{r^2}$$

$$= \frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30} \times 1.9 \times 10^{27}}{(7.8 \times 10^{11})^2}$$

$$= \mathbf{4.145 \times 10^{23}\ N.}$$

The gravitational force of attraction due to the sun provides the necessary centripetal force to the Jupiter to move in the circular orbit. If v is the orbital speed of Jupiter, then

$$\frac{mv^2}{r} = F$$

or
$$v = \sqrt{\frac{rF}{m}} = \sqrt{\frac{7.8 \times 10^{11} \times 4.145 \times 10^{23}}{1.9 \times 10^{27}}}$$

$$= \mathbf{1.304 \times 10^4\ ms^{-1}.}$$

3. Two particles, each of mass m, go round a circle of radius R under the action of their mutual gravitational attraction. Find the speed of each particle.

Solution. The force on each particle is directed along the radius of the circle. The two particles will always lie at the ends of a diameter so that distance between them is $2R$.

$$\therefore \quad F = G\frac{m \times m}{(2R)^2} = \frac{Gm^2}{4R^2}$$

As this force provides the centripetal force, so

$$\frac{Gm^2}{4R^2} = \frac{mv^2}{R}$$

or
$$v = \sqrt{\frac{Gm}{4R}}.$$

4. The mean orbital radius of the earth around the sun is 1.5×10^8 km. Calculate the mass of the sun if $G = 6.67 \times 10^{-11}\ Nm^2kg^{-2}$.

Solution. Here $r = 1.5 \times 10^8$ km $= 1.5 \times 10^{11}$ m

$T = 365$ days $= 365 \times 24 \times 3600$ s

$\because$ Centripetal force required = Force of gravitation between the earth and the sun

$$\therefore \quad \frac{mv^2}{r} = \frac{GMm}{r^2} \quad \text{or} \quad \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{GMm}{r^2}$$

or
$$M = \frac{4\pi^2 r^3}{GT^2}$$

$$= \frac{4 \times 9.87 \times (1.5 \times 10^{11})^3}{6.67 \times 10^{-11} \times (365 \times 24 \times 3600)^2}$$

$$= \mathbf{2.01 \times 10^{30}\ kg.}$$

5. A mass M is broken into two parts of masses m_1 and m_2. How are m_1 and m_2 related so that force of gravitational attraction between the two parts is maximum ?

Solution. Let $m_1 = m$, then $m_2 = M - m$

Force of gravitation between the two parts when they are placed distance r apart is

$$F = G\frac{m(M-m)}{r^2} = \frac{G}{r^2}(Mm - m^2)$$

Differentiating w.r.t m, we get

$$\frac{dF}{dm} = \frac{G}{r^2}(M - 2m)$$

For F to be maximum, $\dfrac{dF}{dm} = 0$

or
$$\frac{G}{r^2}(M - 2m) = 0$$

or $\quad M = 2m \quad$ or $\quad m = M/2$

$\therefore \quad m_1 = m_2 = \mathbf{M/2}.$

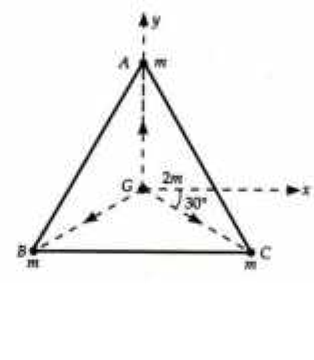
6. Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC, as shown in Fig. 8.6.

(a) What is the force acting on a mass 2m placed at the centroid G of the triangle ?

(b) What is the force if the mass at the vertex A is doubled ?

Take AG = BG = CG = 1 m

Ans - 0 , $2Gm^2$



PROBLEM BASED ON MASS AND DENSITY OF THE EARTH

7. *You are given the following data : $g = 9.81\ ms^{-2}$, $R_E = 6.37 \times 10^6\ m$, the distance to the moon $r = 3.84 \times 10^8\ m$ and the time period of the moon's revolution is 27.3 days. Obtain the mass of the Earth M_E in two different ways.*

Solution. (*i*) $M_E = \frac{gR_E^2}{G} = \frac{9.81 \times (6.37 \times 10^6)^2}{6.67 \times 10^{-11}}$

$= \mathbf{5.97 \times 10^{24}\ kg.}$

(*ii*) From Kepler's law of periods,

$$M_E = \frac{4\pi^2 r^3}{GT^2} = \frac{4 \times 3.14 \times 3.14 \times (3.84)^3 \times 10^{24}}{6.67 \times 10^{-11} \times (27.3 \times 24 \times 60 \times 60)^2}$$

$= \mathbf{6.02 \times 10^{24}\ kg.}$

Both the methods give almost the same mass, the difference being less than 1%.

8. *If the earth were made of lead of relative density 11.3, what then would be the value of acceleration due to gravity on the surface of the earth ? Radius of the earth $= 6.4 \times 10^6\ m$ and $G = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$.*

Solution. Density of the earth,

ρ = Relative density × density of water

$= 11.3 \times 10^3\ kgm^{-3}$

Acceleration due to gravity on the earth's surface,

$$g = \frac{GM}{R^2} = \frac{G}{R^2} \cdot \frac{4}{3}\pi R^3 \times \rho = \frac{4}{3}\pi G R \rho$$

$$= \frac{4}{3} \times \frac{22}{7} \times 6.67 \times 10^{-11} \times 6.4 \times 10^6 \times 11.3 \times 10^3$$

$= \mathbf{22.21\ ms^{-2}}.$

9. *The acceleration due to gravity at the moon's surface is $1.67\ ms^{-2}$. If the radius of the moon is $1.74 \times 10^6\ m$, calculate the mass of the moon. Use the known value of G.*

Solution. Here $G = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$,

$g = 1.67\ ms^{-2}$, $R = 1.74 \times 10^6\ m$

$$M = \frac{g R^2}{G} = \frac{1.67 \times (1.74 \times 10^6)^2}{6.67 \times 10^{-11}} = \mathbf{7.58 \times 10^{22}\ kg.}$$

EXAMPLE 10. *Two lead spheres of 20 cm and 2 cm diameter respectively are placed with centres 100 cm apart. Calculate the attraction between them, given the radius of the earth as 6.37×10^8 cm and its mean density as $5.53 \times 10^3\ kg\ m^{-3}$. Specific gravity of lead = 11.5. If the lead spheres are replaced by brass spheres of same radii, would the force of attraction be same ?*

Solution. Here $r_1 = \frac{20}{2}$ cm $= 0.10$ m,

$r_2 = \frac{2}{2}$ cm $= 0.01$ m

$r = 1.0$ m, $\rho' = 11.5 \times 10^3\ kg\ m^{-3}$

$R = 6.37 \times 10^8$ cm $= 6.37 \times 10^6$ m,

$\rho = 5.53 \times 10^3\ kgm^{-3}$

Masses of the two lead spheres will be

$m_1 = \frac{4}{3}\pi r_1^3 \rho'$

$= \frac{4}{3} \times 3.14 \times (0.10)^3 \times 11.5 \times 10^3 = \mathbf{48.15\ kg}$

$m_2 = \frac{4}{3}\pi r_2^3 \rho'$

The force of attraction between the two lead spheres is

$$F = G\frac{m_1 m_2}{r^2} = \frac{3g}{4\pi R\rho} \times \frac{m_1 m_2}{r^2}$$

$$= \frac{3 \times 9.8 \times 48.15 \times 0.04815}{4 \times 3.14 \times (6.37 \times 10^6) \times 5.53 \times 10^3 \times (1)^2}$$

$= \mathbf{15.4 \times 10^{-11}\ N}$

As the density of brass is less than that of lead, the masses of brass spheres will be smaller than those of lead spheres, so the force of attraction ($F \propto m_1 m_2$) will decrease when lead spheres are replaced by brass spheres.

11. *Compare the gravitational acceleration of the earth due to attraction of the sun with that due to attraction of the moon. Given that mass of sun, $M_s = 1.99 \times 10^{30}$ kg, mass of moon, $M_m = 7.35 \times 10^{22}$ kg, distance of sun from earth, $r_{es} = 1.49 \times 10^{11}$ m and distance of moon from earth $r_{em} = 3.84 \times 10^8$ m.*

Solution. Here $M_s = 1.99 \times 10^{30}$ kg

$M_m = 7.35 \times 10^{22}$ kg

$r_{es} = 1.49 \times 10^{11}$ m

and $r_{em} = 3.84 \times 10^8$ m

Let M_e be mass of earth. If g_{es} is acceleration of the earth due to the attraction of the sun, then

$$M_e\, g_{es} = G\frac{M_e M_s}{r_{es}^2} \quad \text{or} \quad g_{es} = \frac{GM_s}{r_{es}^2} \qquad ...(i)$$

If g_{em} is acceleration of earth due to the attraction of the moon, then

$$M_e\, g_{em} = G\frac{M_e M_m}{r_{em}^2} \quad \text{or} \quad g_{em} = \frac{GM_m}{r_{em}^2} \qquad ...(ii)$$

Dividing equation (*i*) by (*ii*), we get

$$\frac{g_{es}}{g_{em}} = \frac{GM_s}{r_{es}^2} \times \frac{r_{em}^2}{GM_m} = \frac{M_s}{M_m} \times \frac{r_{em}^2}{r_{es}^2}$$

$$= \frac{1.99 \times 10^{30}}{7.35 \times 10^{22}} \times \frac{(3.84 \times 10^8)^2}{(1.49 \times 10^{11})^2} = 179.8$$

12. A body weighs 90 kg f on the surface of the earth. How much will it weigh on the surface of Mars whose mass is 1/9 and the radius is 1/2 of that of the earth ?

Solution. The acceleration due to gravity on the surface of the earth is given by

$$g_e = \frac{GM_e}{R_e^2} \qquad ...(i)$$

The acceleration due to gravity on the surface of Mars is given by

$$g_m = \frac{GM_m}{R_m^2} \qquad ...(ii)$$

Dividing equation (*ii*) by (*i*), we get

$$\frac{g_m}{g_e} = \frac{M_m}{M_e}\left[\frac{R_e}{R_m}\right]^2 = \frac{1}{9}\times\left[\frac{2}{1}\right]^2 = \frac{4}{9}$$

$$\therefore \qquad g_m = \frac{4}{9}\, g_e$$

Weight on the surface of Mars,

$$W_m = mg_m = \frac{4}{9}\, mg_e = \frac{4}{9}\times 90 \text{ kg f} = \mathbf{40 \text{ kg f.}}$$

13. If the radius of the earth shrinks by 2.0%, mass remaining constant, then how would the value of acceleration due to gravity change ? **[Central Schools 09]**

Solution. Acceleration due to gravity on the surface of the earth is given by

$$g = \frac{GM}{R^2}$$

Taking logarithm of both sides, we get

$$\log g = \log G + \log M - 2 \log R$$

As G and M are constant, so differentiation of the above equation gives

$$\frac{dg}{g} = 0 + 0 - 2\,\frac{dR}{R}$$

As radius of the earth decreases by 2%, so

$$\frac{dR}{R} = -\frac{2}{100}$$

$$\frac{dg}{g}\times 100 = -2\,\frac{dR}{R}\times 100$$

$$= -2\times\left(-\frac{2}{100}\right)\times 100 = \mathbf{4\%.}$$

Thus the value of g increases by 4%.

14. A man can jump 1.5 m high on the earth. Calculate the approximate height he might be able to jump on a planet whose density is one-quarter that of the earth and whose radius is one-third of the earth's radius.

Solution. Acceleration due to gravity on the earth's surface is given by

$$g = \frac{4}{3}\pi GR\rho$$

On the planet, $\quad g' = \frac{4}{3}\pi GR'\rho'$

But $\quad R' = \frac{R}{3}, \; \rho' = \frac{\rho}{4}$

$$\therefore \qquad g' = \frac{4}{3}\pi G\times\frac{R}{3}\times\frac{\rho}{4} = \frac{1}{12}\times\frac{4}{3}\pi GR\rho = \frac{1}{12}\, g$$

Assuming that the man puts in the same energy in jumping high on the earth and the planet, then

$$mg'h' = mgh$$

or $\qquad m\times\frac{1}{12}\, g\times h' = mgh$

or $\qquad h' = 12\, h = 12\times 1.5 = \mathbf{18 \text{ m.}}$

PROBLEM BASED ON VARIATION OF 'g' WITH ALTITUDE

15. *At what height from the surface of the earth, will the value of g be reduced by 36% from the value at the surface ? Radius of the earth = 6400 km.*

Solution. Suppose at height h, the value of g reduces by 36% *i.e.*, it becomes 64% of that at the surface. Thus

$$g_h = 64\% \text{ of } g = \frac{64}{100} g$$

But $$g_h = g\frac{R^2}{(R+h)^2}$$

$$\frac{64}{100} g = g\frac{R^2}{(R+h)^2} \quad \text{or} \quad \frac{8}{10} = \frac{R}{R+h}$$

or $$h = \frac{R}{4} = \frac{6400}{4} = \mathbf{1600\ km.}$$

16. *At what height above the earth's surface, the value of g is half of its value on earth's surface ? Given its radius is 6400 km.*

Solution. Here $g_h = g/2$

But $$g_h = g\left(\frac{R}{R+h}\right)^2$$

$\therefore$ $$\frac{g}{2} = g\left(\frac{R}{R+h}\right)^2 \quad \text{or} \quad \left(\frac{R}{R+h}\right)^2 = \frac{1}{2}$$

or $$\frac{R+h}{R} = \sqrt{2}$$

or $$h = (\sqrt{2}-1)\,R = 0.414\ R = 0.414 \times 6400 = \mathbf{2649.6\ km.}$$

17. *Find the percentage decrease in the weight of a body when taken to a height of 32 km above the surface of the earth. Radius of the earth is 6400 km.*

Solution. Here $h = 32$ km, $R = 6400$ km

As $h << R$, so

$$g_h = g\left(1 - \frac{2h}{R}\right) = g - \frac{2gh}{R}$$

or $$g - g_h = \frac{2gh}{R}$$

Percent decrease in weight

$$= \frac{mg - mg_h}{mg} \times 100 = \frac{g - g_h}{g} \times 100$$

$$= \frac{2gh}{g \times R} \times 100 = \frac{2h}{R} \times 100$$

$$= \frac{2 \times 32}{6400} \times 100 = \mathbf{1\%.}$$

18. *A mass of 0.5 kg is weighed on a balance at the top of a tower 20 m high. The mass is then suspended from the pan of the balance by a fine wire 20 m long and is reweighed. Find the change in weight. Assume that the radius of the earth is 6400 km.*

Solution. At a height h ($<< R$), we have

$$g_h = g\left(1 - \frac{2h}{R}\right) = g - \frac{2gh}{R}$$

$\therefore$ $$g - g_h = \frac{2gh}{R}$$

Change in weight = Wt. at the foot of tower − Wt. at the top of the tower

$$= mg - mg_h = m(g - g_h) = \frac{2\ mgh}{R}$$

But $mg = 0.5$ kg f, $h = 20$ m, $R = 6400$ km $= 6.4 \times 10^6$ m

$\therefore$ Change in weight

$$= \frac{2 \times 0.5 \times 20}{6.4 \times 10^6} = \mathbf{3.125 \times 10^{-6}\ kg\ f.}$$

19. *A body hanging from a spring stretches it by 1 cm at the earth's surface. How much will the same body stretch the spring at a place 1600 km above the earth's surface ? Radius of the earth = 6400 km.*

Solution. In equilibrium, weight of the suspended body = Stretching force

$\therefore$ At the earth's surface, $mg = k \times x$

At a height h, $mg' = k \times x'$

$$\frac{g'}{g} = \frac{x'}{x} = \frac{R^2}{(R+h)^2} = \frac{(6400)^2}{(6400+1600)^2}$$

$$= \left(\frac{6400}{8000}\right)^2 = \frac{16}{25}.$$

or $$x' = \frac{16}{25} \times x = \frac{16}{25} \times 1\text{ cm} = \mathbf{0.64\ cm.}$$

PROBLEM BASED ON VARIATION OF 'g' WITH DEPTH

20. *Find the percentage decrease in weight of a body, when taken 16 km below the surface of the earth. Take radius of the earth as 6400 km.*

Solution. Here $R = 6400$ km, $d = 16$ km

$$g_d = g\left(1 - \frac{d}{R}\right) = g\left(1 - \frac{16}{6400}\right) = \frac{399}{400} g$$

$$\therefore \quad g - g_d = g - \frac{399}{400} g = \frac{1}{400} g$$

The percentage decrease in weight of the body

$$= \frac{mg - mg_d}{mg} \times 100 = \frac{g - g_d}{g} \times 100$$

$$= \frac{(1/400)\,g}{g} \times 100 = \mathbf{0.25\%}.$$

21. *How much below the surface of the earth does the acceleration due to gravity become 1% of its value at the earth's surface ? Radius of the earth = 6400 km.*

Solution. Here $g_d = 1\%$ of $g = \frac{g}{100}$

But $\quad g_d = g\left(1 - \frac{d}{R}\right)$

$$\therefore \quad \frac{g}{100} = g\left(1 - \frac{d}{R}\right)$$

or $\quad \frac{d}{R} = 1 - \frac{1}{100} = \frac{99}{100}$

or $\quad d = \frac{99}{100} \times R = \frac{99}{100} \times 6400 = \mathbf{6336\ km.}$

22. *At what height above the earth's surface, the value of g is same as in a mine 80 km deep ?*

Solution. Let h be the height at which 'g' is same as that at depth d. Now

$$g_h = g_d \quad \text{or} \quad g\left(1 - \frac{2h}{R}\right) = g\left(1 - \frac{d}{R}\right)$$

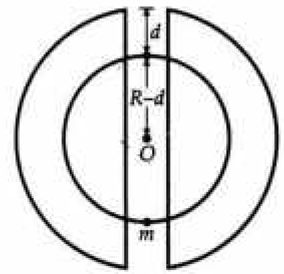
or $\quad \frac{2h}{R} = \frac{d}{R}$

$$\therefore \quad h = \frac{d}{2} = \frac{80}{2} = \mathbf{40\ km.}$$

23. *Imagine a tunnel dug along a diameter of the earth. Show that a particle dropped from one end of the tunnel executes simple harmonic motion. What is the time period of this motion ? Assume the earth to be a sphere of uniform mass density (equal to its known average density $= 5520\ kg\ m^{-3}$.) $G = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$. Neglect all damping forces.*

Solution. The acceleration due to gravity at a depth d below the earth's surface is given by

$$g_d = g\left(1 - \frac{d}{R}\right) \quad \text{or} \quad g_d = g\left(\frac{R-d}{R}\right) = \frac{g}{R} y$$



where $y = R - d =$ Distance of the body from the centre of the earth.

Thus $\quad g_d \propto y$

As the acceleration is proportional to displacement and is directed towards the mean position, so the motion of the body is simple harmonic. Its time period is given by

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{y}{g_d}} = 2\pi\sqrt{\frac{R}{g}}$$

But $\quad g = \frac{GM}{R^2} = \frac{G}{R^2} \times \frac{4}{3}\pi R^3 \rho = \frac{4}{3}\pi G R \rho$

$$\therefore \quad T = 2\pi\sqrt{\frac{R}{\frac{4}{3}\pi G R \rho}} = 2\pi\sqrt{\frac{3}{4\pi G \rho}} = \sqrt{\frac{3\pi}{G\rho}}$$

$$\therefore \quad = \sqrt{\frac{3 \times 3.142}{6.67 \times 10^{-11} \times 5520}} = \mathbf{5059.77\ s.}$$

PROBLEM BASED ON INTENSITY OF GRAVITATIONAL FIELD

1. *Find the intensity of gravitational field when a force of 100 N acts on a body of mass 10 kg in the gravitational field.*

Solution. Here $F = 100$ N, $m = 10$ kg

Intensity of gravitational field,

$$E = \frac{F}{m} = \frac{100 \text{ N}}{10 \text{ kg}} = \mathbf{10\ Nkg^{-1}}.$$

2. *Two bodies of masses 10 kg and 1000 kg are at a distance 1 m apart. At which point on the line joining them will the gravitational field intensity be zero ?*

Solution. Let the resultant gravitational intensity be zero at distance x from the mass of 10 kg on the line joining the centres of the two bodies. At this point, the gravitational intensities due to the two bodies must be equal and opposite.

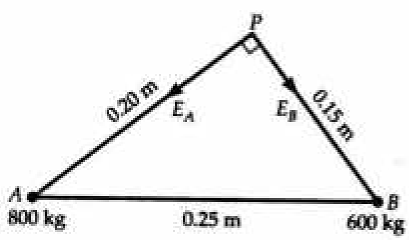
$$\therefore \quad \frac{G \times 10}{x^2} = \frac{G \times 1000}{(1-x)^2}$$

or $\quad 100\,x^2 = (1-x)^2 \quad$ or $\quad 10\,x = 1 - x$

or $\quad 11\,x = 1 \quad$ or $\quad x = \mathbf{1/11\ m.}$

3. *Two masses, 800 kg and 600 kg, are at a distance 0.25 m apart. Compute the magnitude of the intensity of the gravitational field at a point distant 0.20 m from the 800 kg mass and 0.15 m from the 600 kg mass.*

Solution. Let A and B be the positions of the two masses and P the point at which the intensity of the gravitational field is to be computed.



Gravitational intensity at P due to mass at A,

$$E_A = \frac{GM}{r^2} = G\frac{800}{(0.20)^2} = 20{,}000\ G, \text{ along } PA$$

Gravitational intensity at P due to mass at B,

$$E_B = G\frac{600}{(0.15)^2} = \frac{80{,}000}{3}\ G, \text{ along } PB$$

In $\Delta\ APB$,

$$PA^2 + PB^2 = AB^2$$

$$\therefore \quad \angle\ APB = 90°.$$

Hence the magnitude of resultant gravitational intensity at P is

$$E = \sqrt{E_A^2 + E_B^2} = G\sqrt{(20{,}000)^2 + \left(\frac{80{,}000}{3}\right)^2}$$

$$= 6.66 \times 10^{-11} \times \frac{10{,}000}{3} = \mathbf{2.22 \times 10^{-6}\ N.}$$

PROBLEM BASED ON GRAVITATIONAL POTENTIAL

1. *At a point above the surface of the earth, the gravitational potential is -5.12×10^7 J kg^{-1} and the acceleration due to gravity is 6.4 ms^{-2}. Assuming the mean radius of the earth to be 6400 km, calculate the height of this point above the earth's surface.*

Solution. Let r be the distance of the given point from the centre of the earth. Then

Gravitational potential,

$$V = -\frac{GM}{r} = -5.12 \times 10^7 \text{ J kg}^{-1} \quad ...(i)$$

Acceleration due to gravity,

$$g = \frac{GM}{r^2} = 6.4 \text{ ms}^{-2} \quad ...(ii)$$

Dividing (i) by (ii),

$$r = \frac{5.12 \times 10^7}{6.4} = 8 \times 10^6 \text{ m} = 8000 \text{ km}$$

Height of the point from the earth's surface

$= 8000 - 6400 =$ **1600 km.**

2. *The radius of the earth is 6.37×10^6 m, its mean density is 5.5×10^3 kg m^{-3} and $G = 6.66 \times 10^{-11}$ Nm^2 kg^{-2}. Determine the gravitational potential on the surface of the earth.*

Solution. Here $R = 6.37 \times 10^6$ m,

$\rho = 5.5 \times 10^3$ kgm^{-3}, $G = 6.66 \times 10^{-11}$ Nm^2 kg^{-2}

Mass of the earth,

$$M = \text{Volume} \times \text{density} = \frac{4}{3}\pi R^3 \rho$$

Gravitational potential on the earth's surface

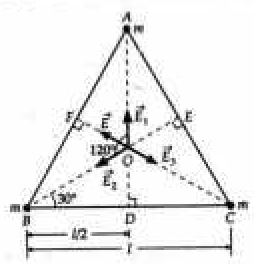
$$V = -\frac{GM}{R} = -\frac{G}{R} \times \frac{4}{3}\pi R^3 \rho = -\frac{4}{3}\pi G R^2 \rho$$

$$= -\frac{4}{3} \times 3.14 \times 6.66 \times 10^{-11} \times (6.37 \times 10^6)^2 \times 5.5 \times 10^3$$

$$= \mathbf{-6.22 \times 10^7 \text{ J kg}^{-1}}.$$

3. *Three mass points each of mass m are placed at the vertices of an equilateral triangle of side l. What is the gravitational field and potential due to three masses at the centroid of the triangle ?*

Solution. In Fig. 8.19, three mass points, each of mass m, are placed at three vertices of equilateral Δ ABC of side l. If O is the centroid of the triangle, then $OA = OB = OC$.



From right ΔODB,

$$\cos 30° = \frac{BD}{OB} = \frac{l/2}{OB}$$

or $$OB = \frac{l/2}{\cos 30°} = \frac{l/2}{\sqrt{3}/2} = \frac{l}{\sqrt{3}}$$

Gravitational fields at O due to mass points at A, B and C are as follows :

$$E_1 = \frac{Gm}{(OA)^2} = \frac{Gm}{(l/\sqrt{3})^2} = \frac{3Gm}{l^2}, \text{ along } \vec{OA}$$

$$E_2 = \frac{Gm}{(OB)^2} = \frac{3Gm}{l^2}, \text{ along } \vec{OB}$$

$$E_3 = \frac{Gm}{(OC)^2} = \frac{3\,Gm}{l^2}, \text{ along } \vec{OC}$$

Angle between $\vec{E}_1$ and $\vec{E}_2$ is 120°. Their resultant is

$$E = \sqrt{E_1^2 + E_2^2 + 2E_1 E_2 \cos 120°}$$

$$= \frac{3Gm}{l^2}\sqrt{1 + 1 + 2 \times \left(-\frac{1}{2}\right)} = \frac{3Gm}{l^2}, \text{ along } \vec{OF}$$

Clearly, $\vec{E}$ is equal and opposite to $\vec{E}_3$, hence the resultant gravitational field at O is zero.

As gravitational potential is a scalar quantity, so the total gravitational potential at O is

$$V = V_1 + V_2 + V_3 = -\frac{Gm}{OA} - \frac{Gm}{OB} - \frac{Gm}{OC}$$

$$= -\frac{3Gm}{OA} = -\frac{3Gm}{l/\sqrt{3}} \quad \left[\because OA = OB = OC = \frac{l}{\sqrt{3}}\right]$$

or $$V = -3\sqrt{3}\,\frac{Gm}{l}.$$

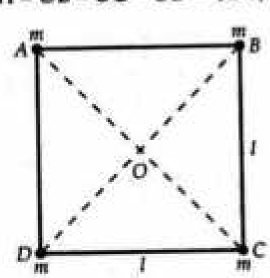
4. *Find the potential energy of a system of four particles, each of mass m, placed at the vertices of a square of side l. Also obtain the potential at the centre of the square.*

Solution.

$$AB = BC = CD = DA = l$$

$$\therefore \quad AC = BD = \sqrt{l^2 + l^2} = \sqrt{2}\,l$$

$$OA = OB = OC = OD = \sqrt{2}\,l/2 = l/\sqrt{2}$$



By the principle of superposition, total potential energy of the system of particles is

$$U = U_{BA} + (U_{CB} + U_{CA}) + (U_{DA} + U_{DB} + U_{DC})$$

$$= 4\,U_{BA} + 2\,U_{DB}$$

$$[\because\ U_{BA} = U_{DA} = U_{DC} = U_{CB},\ U_{CA} = U_{DB}]$$

$$= 4\left(-\frac{Gmm}{l}\right) + 2\left(-\frac{Gmm}{l/\sqrt{2}}\right)$$

$$= -\frac{2\,Gm^2}{l}\left[2 + \frac{1}{\sqrt{2}}\right] = -\frac{2\,Gm^2}{l}[2 + 0.707]$$

$$= -\frac{\mathbf{5.41\,Gm^2}}{\mathbf{l}}$$

Total gravitational potential at the centre O,

$$V = V_A + V_B + V_C + V_D = 4\,V_A = 4\left(-\frac{Gm}{OA}\right)$$

$$= 4\left(-\frac{Gm}{l/\sqrt{2}}\right) = -\frac{\mathbf{4\sqrt{2}\,Gm}}{\mathbf{l}}.$$

5*Two bodies of masses m_1 and m_2 are placed at a distance r apart. Show that at the position where the gravitational field due to them is zero, the potential is given by*

$$V = -\frac{G}{r}[\,m_1 + m_2 + 2\sqrt{m_1 m_2}]$$

Solution. Let the gravitational field be zero at a point P at distance x from m_1 and $(r-x)$ from m_2. Then

$$\frac{Gm_1}{x^2} = \frac{Gm_2}{(r-x)^2} \quad \text{or} \quad \frac{\sqrt{m_1}}{x} = \frac{\sqrt{m_2}}{(r-x)}$$

or $\quad (r-x)\sqrt{m_1} = x\sqrt{m_2} \quad$ or $\quad x(\sqrt{m_1} + \sqrt{m_2}) = r\sqrt{m_1}$

$$\therefore \qquad x = \frac{r\sqrt{m_1}}{\sqrt{m_1} + \sqrt{m_2}} \quad \text{or} \quad \frac{1}{x} = \frac{\sqrt{m_1} + \sqrt{m_2}}{r\sqrt{m_1}}$$

and $\qquad (r-x) = r - \dfrac{r\sqrt{m_1}}{\sqrt{m_1} + \sqrt{m_2}} = \dfrac{r\sqrt{m_2}}{\sqrt{m_1} + \sqrt{m_2}}$

or $\qquad \dfrac{1}{r-x} = \dfrac{\sqrt{m_1} + \sqrt{m_2}}{r\sqrt{m_2}}$

Gravitational potential at point P due to masses m_1 and m_2 will be

$$V = V_1 + V_2 = -\frac{Gm_1}{x} - \frac{Gm_2}{r-x} = -G\left[\frac{m_1}{x} + \frac{m_2}{r-x}\right]$$

$$= -G\left[\frac{m_1(\sqrt{m_1} + \sqrt{m_2})}{r\sqrt{m_1}} + \frac{m_2(\sqrt{m_1} + \sqrt{m_2})}{r\sqrt{m_2}}\right]$$

$$= -\frac{G}{r}[m_1 + m_2 + 2\sqrt{m_1 m_2}].$$

Problem based on escape velocity

1. *Find the velocity of escape at the earth given that its radius is 6.4×10^6 m and the value of g at its surface is $9.8\ ms^{-2}$.*

Solution. Here $R = 6.4 \times 10^6$ m, $g = 9.8\ ms^{-2}$

$$v_e = \sqrt{2gR} = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$

$$= 11.2 \times 10^3\ ms^{-1} = \mathbf{11.2\ kms^{-1}}.$$

2. *Determine the escape velocity of a body from the moon. Take the moon to be a uniform sphere of radius 1.76×10^6 m, and mass 7.36×10^{22} kg. Given $G = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$.*

Solution. Here $R = 1.76 \times 10^6$ m, $M = 7.36 \times 10^{22}$ kg

$$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.36 \times 10^{22}}{1.76 \times 10^6}}$$

$$= 2375\ ms^{-1} = \mathbf{2.375\ km\ s^{-1}}.$$

3. *A black hole is a body from whose surface nothing may even escape. What is the condition for a uniform spherical body of mass M to be a black hole? What should be the radius of such a black hole if its mass is nine times the mass of the earth?* **[Delhi 03C]**

Solution. From Einstein's special theory of relativity, we know that speed of any object cannot exceed the speed of light, $c = 3 \times 10^8\ ms^{-1}$. Thus c is the upper limit to the projectile's escape velocity. Hence for a body to be a black hole,

$$v_e = \sqrt{\frac{2GM}{R}} \le c$$

If $M = 9\ M_E = 9 \times 6 \times 10^{24}$ kg, then

$$R = \frac{2GM}{c^2} = \frac{2 \times 6.67 \times 10^{-11} \times 9 \times 6 \times 10^{24}}{(3 \times 10^8)^2}$$

$$= 8 \times 10^{-2}\ \text{m } \textit{or}\ \textbf{nearly 8 cm.}$$

4. *Jupiter has a mass 318 times that of the earth, and its radius is 11.2 times the earth's radius. Estimate the escape velocity of a body from Jupiter's surface, given that the escape velocity from the earth's surface is $11.2\ km\,s^{-1}$.*

Solution. Escape velocity from the earth's surface is

$$v_e = \sqrt{\frac{2\,GM}{R}} = 11.2\ kms^{-1}$$

Escape velocity from Jupiter's surface will be

$$v'_e = \sqrt{\frac{2GM'}{R'}}$$

But $M' = 318\ M$, $R' = 11.2\ R$

$$\therefore \quad v'_e = \sqrt{\frac{2G(318\ M)}{11.2}} = \sqrt{\frac{2\ GM}{R} \times \frac{318}{11.2}}$$

$$= v_e \times \sqrt{\frac{318}{11.2}} = 11.2 \times \sqrt{\frac{318}{11.2}} = \mathbf{59.7\ kms^{-1}}.$$

5. *Show that the moon would depart for ever if its speed were increased by 42%.*

Solution. The centripetal force required by the moon to revolve around the earth is provided by gravitational attraction.

$$\therefore \quad \frac{mv_0^2}{R} = \frac{GMm}{R^2}$$

or

$$v_0 = \sqrt{\frac{GM}{R}} = \sqrt{\frac{gR^2}{R}} = \sqrt{gR}$$

Velocity required to escape, $v_e = \sqrt{2gR}$

% increase in the velocity of moon

$$= \frac{v_e - v_0}{v_0} \times 100 = \frac{\sqrt{2gR} - \sqrt{gR}}{\sqrt{gR}} \times 100$$

$$= \frac{\sqrt{2} - 1}{1} \times 100 = (1.414 - 1) \times 100 = 41.4\% \simeq \mathbf{42\%}.$$

6. *Calculate the escape velocity for an atmospheric particle 1600 km above the earth's surface, given that the radius of the earth is 6400 km and acceleration due to gravity on the surface of earth is $9.8\ ms^{-2}$.*

Solution. At a height h above the earth's surface, we have

$$v_e = \sqrt{2g_h\,(R + h)}, \quad g_h = \frac{gR^2}{(R + h)^2}$$

$$\therefore \quad v_e = \sqrt{\frac{2 \times gR^2}{(R + h)^2} \times (R + h)} = \sqrt{\frac{2gR^2}{R + h}}$$

But $g = 9.8\ ms^{-2}$, $R = 6.4 \times 10^6$ m,

$h = 1600\ \text{km} = 1.6 \times 10^6$ m,

$$R + h = (6.4 + 1.6) \times 10^6 = 8 \times 10^6\ \text{m}$$

$$\therefore \quad v_e = \sqrt{\frac{2 \times 9.8 \times (6.4 \times 10^6)^2}{8 \times 10^6}}$$

$$= 10.02 \times 10^3\ ms^{-1} = \mathbf{10.02\ kms^{-1}}.$$

7. *The radius of a planet is double that of the earth but their average densities are the same. If the escape velocities at the planet and at the earth are v_p and v_E respectively, then prove that $v_P = 2\ v_E$.*

Solution. If ρ is the average density of the earth, then mass of the earth,

$$M_E = \frac{4}{3}\pi R_E^3 \rho$$

Escape velocity on the earth,

$$v_E = \sqrt{\frac{2G\,M_E}{R_E}} = \sqrt{\frac{2G}{R_E} \times \frac{4}{3}\pi R_E^3 \rho}$$

$$= R_E \sqrt{\frac{8}{3} G \pi \rho}$$

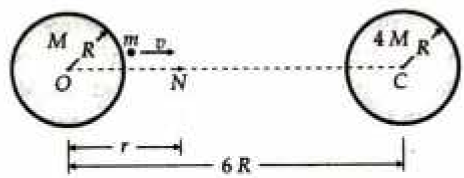
Similarly, escape velocity on the planet,

$$v_P = R_P \sqrt{\frac{8}{3} G \pi \rho}$$

$$\therefore \quad \frac{v_P}{v_E} = \frac{R_P}{R_E}$$

But $\quad R_P = 2\ R_E \quad \therefore \quad v_P = 2\ v_E.$

8. *Two uniform solid spheres of equal radii R, but mass M and 4M have a centre to centre separation 6 R, as shown in Fig. 8.24. The two spheres are held fixed. A projectile of mass m is projected from the surface of the sphere of mass M directly towards the centre of the second sphere. Obtain an expression for the minimum speed v of the projectile so that it reaches the surface of the second sphere.*



Solution. The two spheres exert gravitational forces on the projectile in mutually opposite directions. At the neutral point N, these two forces cancel each other. If $ON = r$, then

$$\frac{GMm}{r^2} = \frac{G(4M)\,m}{(6R - r)^2}$$

or $\quad (6R - r)^2 = 4r^2 \quad$ or $\quad 6R - r = \pm 2r$

or $\quad r = 2R \quad$ or $\quad -6R$

The neutral point $r = -6R$ is inadmissible.

$$\therefore \quad ON = r = 2R$$

It will be sufficient to project the particle m with a minimum speed v which enables it to reach the point N. Thereafter, the particle m gets attracted by the gravitational pull of $4M$.

The total mechanical energy of m at surface of left sphere is

E_i = K.E. of m + P.E. due to left sphere + P.E. due to right sphere

$$= \frac{1}{2} mv^2 - \frac{GMm}{R} - \frac{4\ GMm}{5R}.$$

At the neutral point, speed of the particle becomes zero. The energy is purely potential.

$\therefore \quad E_N$ = P.E. due to left sphere + P.E. due to right sphere

$$= -\frac{GMm}{2R} - \frac{4GMm}{4R}$$

By conservation of mechanical energy,

$$E_i = E_N$$

or $$\frac{1}{2} mv^2 - \frac{GMm}{R} - \frac{4GMm}{5R} = -\frac{GMm}{2R} - \frac{4GMm}{4R}$$

or $$v^2 = \frac{2GM}{R}\left(\frac{4}{5} - \frac{1}{2}\right) = \frac{3GM}{5R}$$

$$\therefore \quad v = \sqrt{\frac{3GM}{5R}}.$$