VISCOSITY AND BERNOULLI'S THEOREM

Viscosity

When a solid body slides over another solid body, the force of friction opposes the relative motion of the solid bodies. In the same way, when a layer of fluid slides over another layer of the same fluid, a force of friction comes into play which is called **viscous force** or **internal force**. This force opposes the relative motion of the fluid.

So, the tendency of fluids to oppose the relative motion of its layers is called **viscosity of fluid**. The backward dragging force called **viscous drag** or **viscous force**.

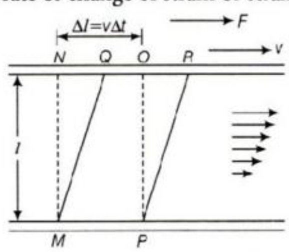
Cause of Viscosity

The velocities of the layers of the liquid increases uniformly from bottom to the top layer. For any layer of liquid, its lower layer pulls it backward while its upper layer pull it forward direction. This type of flow is known as **laminar flow**.

Similar cases arises when the liquid, flowing in a pipe or a tube, then the velocity of the liquid is maximum along the axis of the tube and decreases gradually as it move towards the walls where it becomes zero.

Coefficient of Viscosity

Consider the flow of liquid as shown in figure. A portion of liquid which at some instant having the shape *MNOP* after the short interval of time (say Δt) the fluid is deformed and take the shape as *MQRP* since, the fluid has undergone the shear strain, stress in the solid is the force per unit area but in case of fluid it depends on the rate of change of strain or strain rate.



Strain $\frac{\Delta l}{l}$ and the rate of change of strain is $\frac{\Delta l}{l\Delta t}=\frac{v}{l}$. Hence, the coefficient of viscosity is defined as the ratio of shearing stress to the strain rate

$$\eta=\frac{F/A}{v/l}=\frac{F\cdot l}{v\cdot A}$$

$$\text{Coefficient of viscosity, } \eta=\frac{Fl}{vA}$$

Dimensions of η

As we know $\eta=\frac{F}{A}\cdot\frac{l}{v}$

$$\therefore \quad \eta=\frac{[MLT^{-2}]}{[L^2\,LT^{-1}]}[L]=[ML^{-1}T^{-1}]$$

Units of η

(*i*) In CGS system, the unit of η is **dyne-s/cm^2** and it is called **poise**.

$$1\text{ poise}=\frac{1\text{ dyne}}{1\text{ cm}^2}\times\frac{1\text{cm}}{1\text{ cm/s}}$$

$$\mathbf{1\ poise = 1\ dyne\ s/cm^2}$$

The coefficient of viscosity of a liquid is said to be 1 poise if a tangential force of 1 dyne cm^{-2} of the surface is required to maintain a relative velocity of 1 cm s^{-1} between two layers of the liquid 1 cm apart.

(*ii*) The SI unit of η is **Ns / m^2** or **kg/ms** and it is called **decapoise** or **poiseuille.**

$$1\text{ poiseuille}=\frac{1\text{N}}{1\text{ m}^2}\cdot\frac{1\text{ m}}{1\text{ m/s}}=1\text{ Ns/m}^2\text{ or }P_{a-s}$$

The coefficient of viscosity of a liquid is said to be 1 poiseuille or **decapoise** if a tangential force of 1 Nm^{-2} of the surface is required to maintain a relative velocity of 1 ms^{-1} between two layers of the liquid 1 m apart.

(*iii*) Relation between poiseuille and poise

1 poiseuille or 1 decapoise = 10 poise

The coefficient of viscosity is a scalar quantity.

Relative Viscosity

$$\text{Relative viscosity of liquid}=\frac{\eta_{\text{liquid}}}{\eta_{\text{water}}}$$

Relative viscosity of bloods remains constant between 0° C and 37°C.

Difference between Viscous Force and Solid Friction

	Viscous Force	Solid Friction
1.	Viscous force is directly proportional to the area of layers in contact.	Solid friction is independent of the area of the surfaces in contact.
2.	It is directly proportional to the relative velocity between the two liquid layers.	It is independent of the relative velocity between two solid surfaces.
3.	It is independent of the normal reaction between the two liquid layers.	It is directly proportional to the normal reaction between the surfaces in contact.

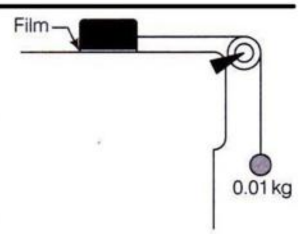
Effect of Temperature on the Viscosity

The viscosity of liquids decreases with increase in temperature and increases with decrease in temperature.

i.e., $\eta\propto\frac{1}{\sqrt{T}}$

On the other hand, the viscosity of gases increases with the increase in temperature and *vice-versa*.

A metal block of area 0.10 m^2 is connected to a 0.01 kg mass via a string that passes over an ideal pulley (considered massless and frictionless), as in figure. A liquid with a film thickness of 0.30 mm is placed between the block and the table. When released the block moves to the right with a constant speed of 0.085 ms^{-1}. Find the coefficient of viscosity of the liquid.



Solution Given, $m = 0.010$ kg, $g = 9.8$ m/s^2, $A = 0.10$ m^2

$$F = T = mg = 0.010 \times 9.8 = 9.8 \times 10^{-2} \text{ N}$$

$$\text{Shear stress on the fluid} = \frac{F}{A} = \frac{9.8 \times 10^{-2}}{0.10} = 0.98 \text{ N/m}^2$$

$$\eta = \frac{\text{Stress}}{\text{Strain rate}}$$

$$\text{Strain rate} = \frac{v}{l} = \frac{0.085}{0.30 \times 10^{-3}}$$

$$\eta = \frac{F}{A}\frac{l}{v} = \frac{9.8 \times 10^{-2} \times 0.30 \times 10^{-3}}{0.10 \times 0.085}$$

$$\eta = 3.45 \times 10^{-3} \text{ Pa-s}$$

Critical Velocity

The critical velocity of a liquid is that limiting value of its velocity of flow upto which the flow is streamlined and above which the flow becomes turbulent.

The critical velocity v_c of a liquid flowing through a tube depends on

(*i*) coefficient of viscosity of the liquid (η)

(*ii*) density of the liquid (ρ)

(*iii*) radius of the tube (r)

Consider, $v_c = k\eta^a \rho^b r^c$

where k is a dimensionless constant. Writing the above equation in dimensional form, we get

$$[M^0LT^{-1}] = [ML^{-1}T^{-1}]^a [ML^{-3}]^b [L]^c$$

$$[M^0LT^{-1}] = [M^{a+b}L^{-a-3b+c}T^{-a}]$$

Compare the powers of M, L and T on both sides, we get

$$a + b = 0$$

$$-a - 3b + c = 1$$

$$-a = -1$$

On solving, we get $a = 1, b = -1, c = -1$

$$\therefore \quad v_c = k\eta\rho^{-1}r^{-1} = \frac{k\eta}{\rho r}$$

We get

$$\boxed{\text{Critical velocity, } v_c = \frac{k\eta}{\rho r}}$$

For the flow to the streamline, value of v_c should be as large as possible. For this η should be large, ρ and r should be small. *So, we conclude that*

(*i*) The flow of liquids of higher viscosity and lower density through narrow pipes tends to be **streamlined**.

(*ii*) The flow of liquids of lower viscosity and higher density through broad pipes tends to become **turbulent** because in that case the critical velocity will be very small.

Stoke's Law

When a small spherical body falls through a viscous fluid at rest, the layers of fluid in contact with the body are dragged alongwith it. But the layers of the fluid away from the body are at rest. This produces a relative motion between different layers of the fluid.

As a result, a **backward dragging** force (*i.e.*, viscous force) comes into play, which opposes the motion of the body. This backward dragging force increases with the increase in velocity of the moving body. Falling of a raindrop, swinging of a pendulum bob are the examples of such type of motion.

Sir George G Stokes (1819–1903), an English scientist found that the backward dragging force F acting on a small spherical body of radius r, moving through a fluid of coefficient of viscosity η, with velocity v is given by

$$\boxed{\text{Dragging force, } F = 6\pi\eta rv}$$

This is called **Stoke's law** of viscosity.

He observed that viscous drag (F) depends upon

(*i*) coefficient of viscosity (η) of the fluid.

(*ii*) velocity (v) of the body.

(*iii*) radius (r) of the spherical body.

Let $F = k\eta^a v^b r^c$ ...(i)

As $[F] = [MLT^{-2}], [\eta] = [ML^{-1}T^{-1}]$

$[v] = [LT^{-1}], [r] = [L]$

So, $[MLT^{-2}] = [ML^{-1}T^{-1}]^a [LT^{-1}]^b [L]^c$

$[MLT^{-2}] = [M^a L^{-a+b+c} T^{-a-b}]$

Compare powers both sides of M, L, T, we get

$$a = 1 \quad \text{...(ii)}$$

$$-a + b + c = 1 \quad \text{...(iii)}$$

$$-a - b = -2 \quad \text{...(iv)}$$

$$\text{or} \quad a + b = 2 \quad \text{...(v)}$$

From Eqs. (iii) and (v), we get $c = 1 \quad b = 1$

Substituting these values in Eq. (i), we get $F = 6\pi\eta rv$

where the value of k was found to be 6π experimentally.

- *This law is used in the determination of electronic charge with the help of Millikan's experiment.*
- *This law accounts the formation of clouds.*
- *This law accounts, why the speed of rain drops is less than that of a body falling freely with a constant velocity from the height of clouds.*
- *This law helps a man coming down with the help of a parachute.*

Terminal Velocity

The maximum constant velocity acquired by a body while falling through a viscous fluid is called its **terminal velocity**.

Consider an example of raindrop in air. It accelerates initially due to gravity. The force of viscosity increases as the velocity of the body increases. A stage is reached, when the true weight of the body becomes just equal to the sum of the upthrust and the viscous force. Then, the body begins to fall with a constant velocity called **terminal velocity**.

When the body attains terminal velocity v, upward thrust + force of viscosity = weight of the spherical body

$$\text{Upward thrust} = Mg = \frac{4}{3}\pi r^3 \sigma g$$

[$\because$ Mass = volume ×density]

$$\text{Similarly weight of the spherical body} = \frac{4}{3}\pi r^3 \rho g$$

$$\frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v = \frac{4}{3}\pi r^3 \rho g$$

where, ρ and σ are the mass densities of sphere and the fluid respectively.

or $$6\pi\eta r v = \frac{4}{3}\pi r^3 (\rho - \sigma) g$$

or $$\boxed{\text{Terminal velocity, } v = \frac{2}{9} \cdot \frac{r^2(\rho - \sigma) g}{\eta}}$$

where, r = radius of the spherical body
v = terminal velocity
η = coefficient of viscosity of fluid
ρ = density of the spherical body
σ = density of fluid.

If 27 drops of rain were to be combine to form one new large spherical drop, then what should be the velocity of this large spherical drop? Consider the terminal velocity of 27 drops of equal size falling through the air is $0.20\ \text{ms}^{-1}$.

Solution Let, the radius of the small drop is r and that of big drop is R. The volume of the big drop = 27 × volume of each small drop

$$\frac{4}{3}\pi R^3 = 27 \times \frac{4}{3}\pi r^3 \Rightarrow R = 3r$$

Let, the terminal velocities of small and big drop are v_1 and v_2 respectively. Then,

$$v = \frac{2r^2(\rho - \sigma) g}{9\eta} \Rightarrow v \propto r^2$$

Hence, $$\frac{v_2}{v_1} = \frac{R^2}{r^2}$$

$$\Rightarrow \quad v_2 = v_1 \times \frac{R^2}{r^2} = 0.2\left(\frac{3r}{r}\right)^2 = 0.2 \times 9$$

$$v_2 = 1.8\ \text{m/s}$$

Reynold's Number

Osborne Reynolds (1842-1912) observed that turbulent flow is less likely for viscous fluid flowing at low rates. He defined a dimensionless parameter whose value decides the nature of flow of a liquid through a pipe, *i.e.,* whether a flow will be steady or turbulent, it is given by

$$\boxed{\text{Reynold's number, } R_e = \frac{\rho v D}{\eta}}$$

where, ρ = density of the liquid
v = velocity of the liquid
η = coefficient of viscosity of the liquid
D = diameter of the pipe.

(*i*) If R_e lies between 0 and 2000, then liquid flow is streamline or laminar.
(*ii*) If $R_e > 3000$, then liquid flow is turbulent.
(*iii*) If R_e lies between 2000 and 3000, then flow of liquid is unstable, it may change from laminar to turbulent and *vice-versa*.

The exact value at which turbulent sets in a fluid is called **critical Reynold's number**.

In another form R_e can also be written as,

$$R_e = \frac{\rho v D}{\eta} = \frac{\rho v^2}{\left(\eta \frac{v}{D}\right)} = \frac{\rho A v^2}{\eta \frac{Av}{D}} = \frac{\text{Intertial force}}{\text{Force of viscosity}}$$

Hence, Reynold's number represents the ratio of the inertial force per unit area to the viscous force per unit area.

The flow rate of water is 0.58 L/mm from a tap of diameter of 1.30 cm. After sometime the flow rate is increased to 4 L/min. Determine the nature of the flow for both the flow rates. The coefficient of viscosity of water is 10^{-3} Pa - s *and the density of water is* $10^3\ \text{kg/m}^3$.

Solution Given, diameter $D = 1.30\ \text{cm} = 1.3 \times 10^{-2}\ \text{m}$

Coefficient of viscosity of water $\eta = 10^{-3}\ \text{Pa-s}$

Density of water, $\rho = 10^3\ \text{kg/m}^3$

The volume of the water flowing out per second is

$$V = vA = v \times \pi r^2 = \pi \frac{D^2}{4}$$

$\therefore$ Speed of flow $v = \frac{4V}{\pi D^2}$

Reynold's number, $R_e = \frac{\rho v D}{\eta} = \frac{4\rho V}{\eta \pi D}$

Case I When $V = 0.58 \text{ L/min} = \frac{0.58 \times 10^{-3} \text{ m}^3}{1 \times 60 \text{ s}}$

$$= 9.67 \times 10^{-6} \text{ m}^3\text{s}^{-1}$$

$$R_e = \frac{4 \times 10^3 \times 9.67 \times 10^{-6}}{10^{-3} \times 3.14 \times 1.3 \times 10^{-2}} = 948$$

$\because$ $R_e < 1000$, so the flow is steady or streamline

Case II When $V = 4 \text{ L/min}$

$$= \frac{4 \times 10^{-3}}{60} \text{ m}^3\text{s}^{-1} = 6.67 \times 10^{-5} \text{ m}^3\text{s}^{-1}$$

$$R_e = \frac{4 \times 10^3 \times 6.67 \times 10^{-5}}{10^{-3} \times 3.14 \times 1.3 \times 10^{-2}} = 6539$$

$\because$ $R_e > 3000$, so the flow will be turbulent.

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1 . *In Millikan's oil drop experiment, what is the terminal speed of an uncharged drop of radius 2.0×10^{-5} and density 1.2×10^3 kg m^{-3}. Take the viscosity of air at the temperature of the experiment to be 1.8×10^{-5} Pa s. How much is the viscous force on the drop at that speed ? Neglect buoyancy of the drop due to air.*

Sol. Given $r = 2.0 \times 10^{-5}$ m

$\rho = 1.2 \times 10^3$ kg/m³

$\eta = 1.8 \times 10^{-5}$ Pa-s.

We know that

Terminal velocity, $v = \frac{2r^2(\rho - \rho_0)g}{9\eta}$

Putting respective values

$$v = \frac{2 \times (2.0 \times 10^{-5})^2 (1.2 \times 10^3 - 0) \times 9.8}{9 \times 1.8 \times 10^{-5}}$$

$$\boxed{v = 5.8 \times 10^{-2} \text{ ms}^{-1}}$$

Viscous force on the drop;

$F = 6\eta\pi. rv$

$$= 6 \times \frac{22}{7} \times (1.8 \times 10^{-5}) \times (2.0 \times 10^{-5}) \times (5.8 \times 10^{-2})$$

$$\boxed{F = 3.93 \times 10^{-10} N}$$

2 . *(a) What is the largest average velocity of blood flow in an artery of radius 2×10^{-3} m if the flow must remain laminar ? (b) What is the corresponding flow rate ? (Take viscosity of blood to be 2.084×10^{-3} Pa s).*

Sol. Given $r_{artery} = 2 \times 10^{-3}$ m

flow is laminar so, $N_R = 2000$

(*a*) $v_{max} = \frac{N_R n}{\rho D} = \frac{2000 \times 2.084 \times 10^{-3}}{(1.06) \times 10^3 \times 4 \times 10^{-3}}$

$v_{max} = 0.98 \text{ ms}^{-1}$

(*b*) Volume flowing per sec.

$V = \pi r^2 v_{max}$

$= 3.14 \times (2 \times 10^{-3})^2 \times 0.98$

$= 1.23 \times 10^{-5} \text{ m}^3 \text{ s}^{-1}$

$\Rightarrow$ $\boxed{V = 1.23 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}}$

Hydrodynamics

We have studied the fluids at rest or hydrostatics. Now, we will learn the branch of physics which deals with the study of fluids in motion called **fluid dynamics** or **hydrodynamics**.

Streamline

Streamline flow of a liquid is that flow in which each particle of the liquid passing through a point travels along the same path and with the same velocity as the preceeding particle passing through the same point.

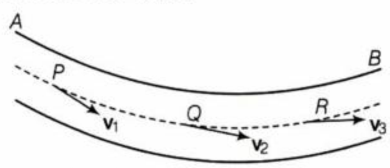
It is also defined as a curve whose tangent at any point is in the direction of the fluid velocity at that point.

Trajectory of a fluid particle

Properties of Streamline

(*i*) In streamline flow, no two streamlines can cross each other. If they do so, the particles of the liquid at the point of intersection will have two different directions for their flow, which will decrease the steady nature of the liquid flow.

(*ii*) The greater is the crowding of streamline at a place greater is the velocity of the liquid particles at that place and *vice-versa*.

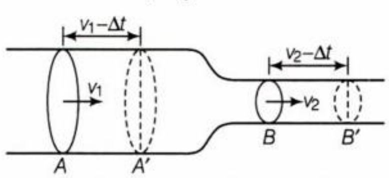


A region of streamline flow

Equation of Continuity

It states that, *during the streamline flow of the non-viscous and incompressible fluid through a pipe of varying cross-section, the product of area of cross-section and the normal fluid velocity* (av) *remains constant throughout the flow.*

Consider a non-viscous and incompressible liquid flowing through a tube AB of varying cross-section.



Fluid flow through pipe AB

Let a_1 be the area of cross-section, v_1 fluid velocity, ρ_1 fluid density at section A and the values of corresponding quantities at section B be a_2, v_2 and ρ_2.

As, mass = volume × density

= Area of cross-section × length × density

$\therefore$ Mass of fluid that entering through section A in time Δt,

$$m_1 = a_1 v_1 \Delta t \rho_1$$

Mass of fluid that leaving through section B in time Δt,

$$m_2 = a_2 v_2 \Delta t \rho_2$$

According to conservation of mass, we get

$$m_1 = m_2$$

or $$a_1 v_1 \Delta t \rho_1 = a_2 v_2 \Delta t \rho_2$$

As the fluid is incompressible, then $\rho_1 = \rho_2$ and hence

$$a_1 v_1 = a_2 v_2$$

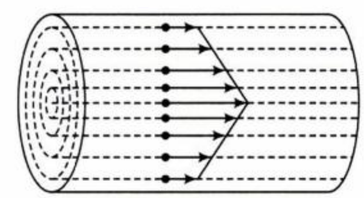
or $$\boxed{av = \text{constant}}$$

This is known as **equation of continuity**.

Laminar Flow

If the liquid flows over a horizontal surface in the form of layers of different velocities, then the flow of liquid is called **laminar flow**.

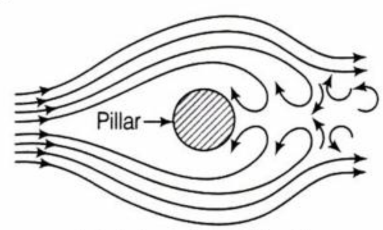
In laminar flow, the particle of one layer do not enter into another layer. In general, laminar flow is a streamline flow as shown in figure.


Laminar flow of liquid

Turbulent Flow

In rivers and canals where speed of water is quite high or the boundary surfaces cause abrupt changes in velocity of the flow, then the flow becomes irregular. Such flow of liquid is known as **turbulent flow**.

Thus, the flow of fluid in which velocity of all particles crossing a given point is not same and the motion of the fluid becomes irregular or disordered is called **turbulent flow** as shown in figure.



Turbulent flow of liquid

Ideal Fluid

The motion of real fluids is very complicated. To understand fluid dynamics in a simpler manner, we assume that the fluid is ideal. An ideal fluid is one which is non-viscous, incompressible, and its flow is steady and irrotational.

Bernoulli's Theorem

Bernoulli's principle is based on the law of conservation of energy and applied to ideal fluids. It states that

the sum of pressure energy per unit volume, kinetic energy per unit volume and potential energy per unit volume of an incompressible, non-viscous fluid in a streamlined irrotational flow remains constant at every cross-section throughout the liquid flow.

Mathematically, it can be expressed as

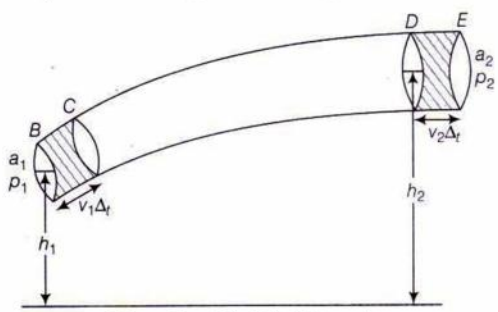
$$\boxed{p + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}}$$

where, p represents for pressure energy per unit volume $\frac{1}{2}\rho v^2$ for kinetic energy per unit volume and $\rho g h$ for potential energy per unit volume.

The Swiss physicist **Daniel Bernoulli** developed this relationship in 1738.

Proof Consider an ideal fluid having streamline flow through a pipe of varying area of cross-section as shown in figure.

Let p_1, a_1, h_1, v_1 and p_2, a_2, h_2, v_2 be the pressure, area of cross-section, height and velocity of flow at points A and B respectively. Force acting on fluid at point $A = p_1 a_1$



Flow of an ideal fluid in a pipe

Distance travelled by fluid in one second at point A

$$= v_1 \times 1 = v_1$$

Work done per second on the fluid at point A

= Force × distance travelled by fluid in one second.

$$A = p_1 a_1 \times v_1$$

Similarly, work done per second by the fluid at point B

$$= p_2 a_2 v_2$$

∴ Net work done on the fluid by pressure energy,

$$W = p_1 a_1 v_1 - p_2 a_2 v_2$$

But $a_1 v_1 = a_2 v_2 = \frac{m}{\rho}$ (*i.e.*, equation of continuity)

∴ Net work done on the fluid by the pressure energy,

$$W = \left(\frac{p_1 m}{\rho} - \frac{p_2 m}{\rho}\right)$$

Total work done by the pressure energy on the fluid increases the kinetic energy and potential energy of the fluid when it flows from A to B.

∴ Increase in potential energy of fluid

$$= \text{KE at } B - \text{KE at } A$$

$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \qquad [v_2 > v_1 \text{ and } a_1 > a_2]$$

Similarly, total increase in potential energy $= mgh_2 - mgh_1$

According to work energy theorem work done by the fluid is equal to change in the energy of fluid.

i.e., work done by the pressure energy = total increase in energy

$$\therefore \frac{p_1 m}{\rho} - \frac{p_2 m}{\rho} = (mgh_2 - mgh_1) + \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right)$$

$$\left(\frac{p_1 - p_2}{\rho}\right) = (gh_2 - gh_1) + \frac{1}{2}v_2^2 - \frac{1}{2}v_1^2$$

or $$\frac{p_1}{\rho} + \frac{1}{2}v_1^2 + gh_1 = \frac{p_2}{\rho} + \frac{1}{2}v_2^2 + gh_2$$

$$\therefore \qquad \rho = \frac{m}{V}$$

Hence, $$\frac{p}{\rho} + gh + \frac{1}{2}v^2 = \text{constant}$$

$$\boxed{p + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}} \qquad \ldots\text{(i)}$$

Dividing both sides of Eq. (i) by ρg, we get

$$\frac{p}{\rho g} + h + \frac{v^2}{2g} = \frac{\text{constant}}{\rho g} = \text{new constant} \qquad \ldots\text{(ii)}$$

Here, $\frac{p}{\rho g}$ is called pressure head, h is called gravitational head and $\frac{v^2}{2g}$ is called velocity head.

Equation (ii) enable us to state Bernoulli's theorem in the streamline flow of an ideal liquid as the sum of pressure head, gravitational head and velocity head is always constant.

If the fluid is flowing through a horizontal tube, two ends of the tube are at the same level. Therefore, there is no gravitational head (level difference) *i.e.*, $h = 0$.

$$\frac{p}{\rho} + \frac{1}{2}v^2 = p + \frac{1}{2}\rho v^2 = \text{constant}$$

This shows if **p increases, v decreases and *vice-versa*.** Thus, Bernoulli's theorem also states that in the streamline flow of an ideal liquid through a horizontal tube, the velocity increases where pressure decreases and *vice-versa*. This is also called **Bernoulli's principle**.

Bernoulli's Equation for the Fluid at Rest

When a fluid at rest *i.e.,* the velocity is zero everywhere, then the Bernoulli's equation becomes

$$p_1 + \rho g h_1 = p_2 + \rho g h_2$$

$$\boxed{p_1 - p_2 = \rho g (h_2 - h_1)}$$

Limitations of Bernoulli's Theorem

(*i*) Bernoulli's equation ideally applies to fluids with zero viscosity or non-viscous fluids.

(*ii*) The fluids must be incompressible, as the elastic energy of the fluid is also not taken into consideration.

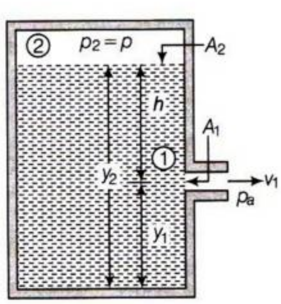
(*iii*) Bernoulli's equation is applicable only to streamline flow of a fluid. It is not valid for non-steady or turbulent flow.

Applications of Bernoulli's Theorem

(i) Speed of Efflux (Torricelli's Law)

According to Torricelli's, velocity of efflux *i.e.*, the velocity with which the liquid flows out of an orifice (*i.e.*, a narrow hole) is equal to that which a freely falling body would acquire in falling through a vertical distance equal to the depth of orifice below the free surface of liquid.

Speed of efflux The word efflux means the outflow of a fluid as shown in figure. Consider a tank containing a liquid of density ρ with a small hole on its side at height y_1 from the bottom and y_2 be the height of the liquid surface from the bottom and p be the air pressure above the liquid surface.



If A_1, A_2 are the cross-sectional areas and v_1, v_2 are the velocities of liquid at point 1 and 2, then from the equation of continuity we get

$$A_1 v_1 = A_2 v_2 \quad \text{or} \quad v_2 = \frac{A_1}{A_2} v_1$$

If $A_2 >> A_1$ so the liquid may be taken at rest at the top *i.e.*, $v_2 \approx 0$.

Applying the Bernoulli's theorem at points 1 and 2. The pressure $p_1 = p_a$ (the atmospheric pressure)

We get $$p_a + \frac{1}{2}\rho v_1^2 + \rho g\, y_1 = p + \rho g\, y_2 \quad [\because\ v_2 \approx 0]$$

or $$\frac{1}{2}\rho v_1^2 = pg\,(y_2 - y_1) + (p - p_a)$$

$$y_2 - y_1 = h$$

Hence, $$\frac{1}{2}\rho v_1^2 = \rho g h + (p - p_a)$$

Velocity of the liquid falls from orifice, $$\boxed{v_1 = \sqrt{2gh + \frac{2(p - p_a)}{\rho}}}$$

Special conditions (*i*) If $p >> p_a$, $2gh$ is neglected

$$v_1 = \sqrt{\frac{2(p - p_a)}{\rho}}$$

(*ii*) When the tank is open to the atmosphere,

$$p = p_a \quad \text{and} \quad v_1 = \sqrt{2gh}$$

Thus, the **velocity of efflux** of a liquid is equal to the velocity which a body acquires in falling freely from the free liquid surface to the orifice. This result is called **Torricelli's law**.

(ii) Venturimeter

It is a device used to measure the flow speed of incompressible fluid. It is also called **flow meter** or **venturi tube**.

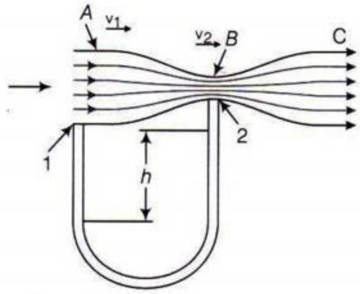
Construction It consists of a horizontal tube having wider opening of cross-section a_1 and a narrow neck of cross-section a_2.

These two regions of the horizontal tube are connected to a manometer, containing a liquid of density ρ_m.

Working Let the liquid velocities be v_1 and v_2 at the wider and narrow region of the tube respectively. Let p_1 and p_2 are liquid pressures at region A and B then,

According to the equation of continuity,

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad \frac{a_1}{a_2} = \frac{v_2}{v_1}$$



Using Bernoulli's equation for horizontal flow of liquid with density ρ.

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

or
$$p_1 - p_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$$

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2\left(\frac{v_2^2}{v_1^2} - 1\right)$$

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2\left(\frac{a_1^2}{a_2^2} - 1\right)$$

This pressure difference cause the liquid in the arm 2 of U tube connected at the narrow tube B to rise in comparison to other arm 1. The difference in height h of two arms of U tube measures the pressure difference.

$$p_1 - p_2 = h\rho_m g$$

$$h\rho_m g = \frac{1}{2}\rho v_1^2\left(\frac{a_1^2}{a_2^2} - 1\right)$$

from above equation

$$\text{Velocity of flow, } v_1 = \sqrt{\frac{2h\rho_m g}{\rho}}\left(\frac{a_1^2}{a_2^2} - 1\right)^{-1/2}$$

It is speed of liquid in the wider tube.

The volume of the liquid flowing per second through the wider tube is

$$V = a_1 v_1 = a_1\sqrt{\frac{2h\rho_m g}{\rho}}\left(\frac{a_1^2}{a_2^2} - 1\right)^{-1/2}$$

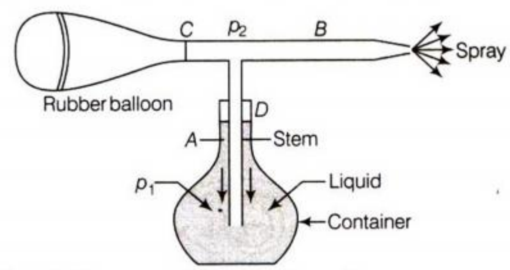
$$= a_1 a_2\sqrt{\frac{2h\rho_m g}{\rho(a_1^2 - a_2^2)}} = a_1 a_2\sqrt{\frac{2(p_1 - p_2)}{\rho(a_1^2 - a_2^2)}}$$

So,
$$V = a_1 a_2\sqrt{\frac{2(p_1 - p_2)}{\rho(a_1^2 - a_2^2)}}$$

(iii) Atomizer or Sprayer

It is based on the Bernoulli's principle. The essential parts of an atomizer are shown in figure. The forward stroke of the piston produces a stream of air past the end of the tube D is immersed in the liquid to be sprayed. The air flowing past the open end of the tube reduces the pressure on the liquid.

So, the atmospheric pressure acting on the surface of liquid, forces the liquid into the tube D. As a result the liquid rises up in the vertical tube A. When it collides with the high speed air in tube B, it breaks up into fine spray.



(iv) Blood Flow and Heart Attack

Consider a case where a person suffering from heart attack problem, whose artery gets constricted due to the accumulation of plaque on its inner walls.

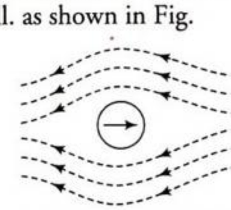
According to Bernoulli's principle, the pressure inside artery becomes low and the artery may collapse due to external pressure. The activity of heart is further increased in order to force the blood through that artery. As the blood rushes through that artery, the internal pressure once again drops due to same reasons. This will be leading to a repeat collapse. This phenomenon is called **vascular flutter** which can be heard on a stethoscope. This may result in a heart attack.

(v) Dynamic Lift

Dynamic lift is the force that acts on a body by virtue of its motion through a fluid. It is responsible for the curved path of a spinning ball and the lift of an aircraft wing.

(a) Ball Moving without Spin

When the velocity of the air above the ball is same as below the ball at the corresponding points resulting in zero pressure difference. The air therefore exerts no upward or downward force on the ball. as shown in Fig.

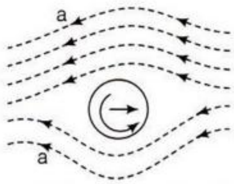


Ball moving without spin

(b) Ball Moving with Spin

As the ball moves to the right, air rushes to **the left with** respect to the ball. Since the ball is spinning, it **drags some air** with it because of the roughness of its surface. **The speed of** air above the ball with respect to it is greater **than below the** ball. Hence, the pressure below the ball is **greater than that** above the ball. The force acts on the ball which makes it follow a curved path, as shown in Fig.

The difference in lateral pressure, which causes a spinning ball to take a curved path which is curved towards the greater pressure side, is called **magnus effect**.



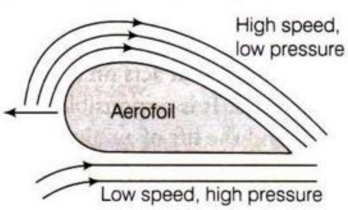
Ball moving with spin

(c) Aerofoil, Lift of an Aircraft Wing

Aerofoil is a solid object shaped to provide an **upward dynamic lift** as it moves horizontally through air. This upward force makes aeroplane fly. The cross-section of the wing of an aeroplane looks like an aerofoil.

When the aeroplane moves through air, the air in the region above the wing moves faster than the air below as seen from the streamlines above the wing.

The difference in speed in the two regions makes the pressure in the region above lower than the pressure below the wing producing thereby a dynamic lift.



Aerofoil

Roof can Blown off Without Damaging the House

During wind storm, the roofs of some huts are blown off without damaging the other parts of the house. The high wind blowing over the roof creates a low pressure p_2 in accordance with Bernoulli's principle.

The pressure p_1 below the roof is equal to the atmospheric pressure which is larger than p_2. This difference of pressure provides a vertical lift to the roof of hut. When lift is sufficient to overcome the gravity pull on the roof, the roof of the hut is blown off without causing any damange to the walls of hut.

1. *Can Bernoulli's equation be used to describe the flow of water through a rapid in a river? Explain.*

Sol. No, Bernoulli's theorem is used for streamline flow only.

2. *Does it matter if one uses gauge instead of absolute pressures in applying Bernoulli's equation? Explain.*

Sol. No, it doesn't matter if one uses gauge instead of absolute pressures in applying Bernoulli's equation, unless the pressure (atmospheric) at the two points where Bernoulli's equation is applied are significantly different.

3. *In a test experiment on a model aeroplane in a wind tunnel, the flow speeds on the upper and lower surfaces of the wind are 70 ms^{-1} and 63 ms^{-1} respectively. What is the lift on the wing if its area is 2.5 m^2? Take the density of air to be 1.3 kg m^{-3}.*

Sol. Let V_U and V_L be the speeds on the upper and lower surfaces of the wing of the aeroplane and P_U and P_L be the pressure on the upper and lower surfaces of the wing respectively.

Then, $V_U = 70\ \text{ms}^{-1}$

$V_L = 63\ \text{ms}^{-1}$

$\rho_{air} = 1.3\ \text{kg m}^{-3}$.

Applying Bernoulli's theorem

$$\frac{P_U}{\rho} + gh + \frac{1}{2}V_U^2 = \frac{P_L}{\rho} + gh + \frac{1}{2}V_L^2$$

$$\Rightarrow \quad \frac{P_U - P_L}{\rho} = \frac{1}{2}\left(V_U^2 - V_L^2\right)$$

$$\Rightarrow \quad P_U - P_L = \frac{1}{2}\rho(V_U^2 - U_L^2)$$

$$= \frac{1}{2} \times 1.3\left[(70)^2 - (63)^2\right]$$

$$= 605.15\ \text{Pa.}$$

This difference in pressure provides the necessary lift to the aeroplane.

So, lift of the aeroplane,

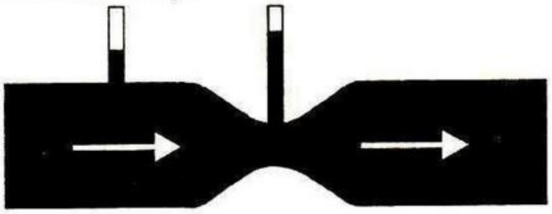
T = Pressure difference × area of the wings

$T = 605.15 \times 2.5$

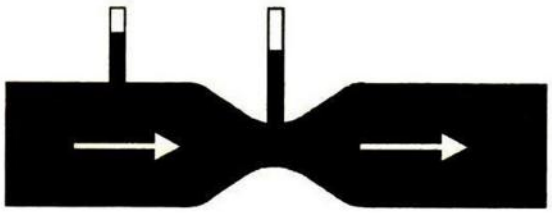
$T = 1512.875$ N

$$\boxed{T = 1.51 \times 10^3\ \text{N}}$$

4. *Fig. (a) and (b) refer to the steady flow of a (non-viscous) liquid. Which of the two figures is incorrect ? Why ?*



(a)



(b)

Sol. Fig. (*a*) is incorect.

According to the equation of continuity,

av = constant,

When area of the cross-section of the tube is less, the velocity of the liquid flow is more. So velocity of the liquid is more at the constriction, than the other portions of the tube.

According to Bernoulli's theorem

$$P + \frac{1}{2}\rho v^2 = \text{a constant.}$$

$\Rightarrow$ when v is less, pressure 'P' is more and vice-versa.

5. *The cylindrical tube of a spray pump has a cross-section of 8.0 cm² one end of which has 40 fine holes each of diameter 1.0 mm. If the liquid flow inside the tube is 1.5 m min⁻¹, what is the speed of ejection of the liquid through the holes ?*

Sol. Given $a_1 = 8.0 \text{ cm}^2$

$a_1 = 8 \times 10^{-4} \text{ m}^2$

n = no. of holes = 40

D = diameter of each hole

= 1 mm

= 10^{-3} m.

$\therefore$ radius of hole = $\frac{D}{2} = 5 \times 10^{-4}$ m

Area of cross-section of each hole

$a = \pi r^2$

$= \pi(5 \times 10^{-4})^2 \text{ m}^2$

Total area of cross-section of 40 holes

$A = 40 \times \pi(5 \times 10^{-4})^2 \text{ m}^2$

Speed of liquid inside the tube

$v_1 = 1.5$ m/min.

$= \frac{1.5}{60} \text{ ms}^{-1}$ **Ans.**

If v_2 is the velocity of ejection of the liquid through the holes, then

$a_1 v_1 = a_2 v_2$

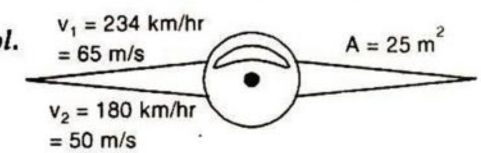
$\Rightarrow \quad v_2 = (a_1 v_1)/A$

$$\Rightarrow \quad v_2 = \frac{(8 \times 10^{-4}) \times 1.5}{60 \times 40 \times \pi \times (5 \times 10^{-4})^2}$$

$v_2 = 0.637 \text{ ms}^{-1}$. **Ans.**

6. *A plane is in level flight at constant speed and each of its two wings has an area of 25 m². If the speed of the air is 180 km/h over the lower wing and 234 km/h over the upper wing surface, determine the plane's mass. (Take air density to be 1 kg m⁻³).*

Sol.



Front view of the flying aeroplane

Applying Bernoulli's theorem

$$\therefore P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\text{or } P_2 - P_1 = \frac{1}{2}\rho(v_1^2 - v_2^2) = \frac{1}{2} \times 1(65^2 - 50^2)$$

$$= \frac{1}{2}(65 - 50)(65 + 50)$$

$$= \frac{1}{2} \times 15 \times 115 = 862.5 \text{ N/m}^2$$

$\therefore$ Upward force on two wings

$= 862.5 \times (25 \times 2)$ N.

Hence, this force will support the weight (mg) of the aeroplane.

$\therefore \quad m \times 9.8 = 862.5 \times 25 \times 2$

$m = 4400$ kg. **Ans.**

7. *A non-viscous liquid of constant density 1000 kgm^{-3} flows in a streamline motion along a tube of variable cross-section. The tube a kept inclined in the vertical plane as shown in figure. The area of cross-section of the tube at two points P and Q at heights of 2 m and 5 m are respectively $4 \times 10^{-3} m^2$ and $8 \times 10^{-3} m^2$. The velocity of the liquid at point P is 1 ms^{-1}. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from point P to Q.*

Given, $\rho = 1000\ kg/m^3, v_1 = 1\ m/s,\ a_1 = 4 \times 10^{-3}\ m^2,$
$a_2 = 8 \times 10^{-3}\ m^2,\ h_1 = 2\ m,\ h_2 = 5\ m$

Sol. Apply Bernoulli's theorem,

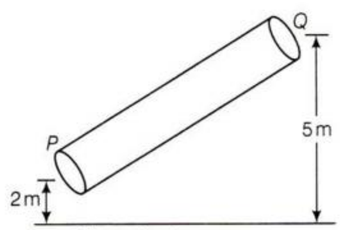
$$p_1 + \frac{1}{2}\rho v_1^2 + g\rho h_1 = p_2 + \frac{1}{2}\rho v_2^2 + g\rho h_2$$

$$(p_1 - p_2) = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1)$$

where,

$(p_1 - p_2)$ = Work done by pressure per unit volume

$$i.e., \left(\frac{W}{\text{Volume}}\right)_p = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) \quad \ldots(i)$$



From equation of continuity,

$$a_1 v_1 = a_2 v_2$$

$$v_2 = \frac{a_1 v_1}{a_2} = \frac{4 \times 10^{-3} \times 1}{8 \times 10^{-3}} = 0.5\ m/s$$

$$\left(\frac{W}{\text{Volume}}\right)_p = \frac{1}{2} \times 1000\,[0.25 - 1] + 1000 \times 10\,(5 - 2)$$

$$= -375 + 30{,}000 = 29625\ J/m^3$$

Work done per unit volume by the gravitational force

$$= \rho g(h_1 - h_2)$$

$$= 1000 \times 10\,(2 - 5) = -3 \times 10^4\ J/m^3$$

8. *Explain why?*

(i) To keep a piece of paper horizontal, you should blow over, not under it.

(ii) When we try to close a water tap with our fingers, fast jets of water gush through the opening between our fingers.

(iii) The size of the needle of a syringe controls flow rate better than the thumb pressure exerted by a doctor while administering an injection.

(iv) A fluid flowing out of a small hole in a vessel results in a backward thrust on the vessel.

(v) A spinning cricket ball in air does not follow a parabolic trajectory.

According to Bernoulli's theorem, for horizontal flow of fluids,

$$p + \frac{1}{2}\rho v^2 = \text{constant}$$

Therefore, when velocity of a fluid increases, its pressure decreases and vice-versa.

Sol. (*i*) When we blow over a piece of paper, the velocity of air above the paper increases. So, in accordance with Bernoulli's theorem $\left(p + \frac{1}{2}\rho v^2 = \text{constant}\right)$, the pressure of air decreases above the paper. Due to the pressure difference of a air between below and above the paper a lifting force acts on paper and hence it remain horizontal.

(*ii*) According to equation of continuity, for steady flow of liquid the product of area of cros -section of the tube and velocity of liquid remains constant at each point

i.e., $A_1 v_1 = A_2 v_2$

When we try to close a water tap with our fingers, the area of cross-section of the outlet of water jet is reduced due to the restriction provided by the fingers and therefore, the velocity of water increases and fast jets of water gush through the openings between our fingers.

(*iii*) According to Bernoulli's theorem,

$$p + \frac{1}{2}\rho v^2 = \text{constant}$$

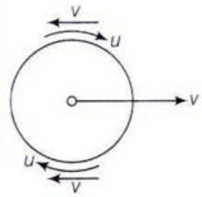
In this relation, pressure (p) occurs with one power while velocity (v) occurs with two powers. Therefore, the influence of velocity is higher than pressure. The size of the needle controls the velocity of flow and the thumb pressure controls pressure. Therefore, size of the needle of a syringe controls flow rate better than thumb pressure.

(*iv*) A fluid flowing out of a small hole in a vessel have a large velocity and therefore a large momentum. As no external force is acting, therefore according to law of conservation of momentum equal momentum in opposite direction and hence, a backward velocity is attained by the vessel. Therefore, a backward thrust $\left(F = \frac{dp}{dt}\right)$ acts on the vessel.

(*v*) A spinning cricket ball in air does not follow a parabolic trajectory due to Magnus effect.

Let a spinning cricket ball is moving forward with a velocity v and spinning clockwise with velocity u. As ball moves forward, if leaves a, lower pressure region behind it.

To fill this region, air moves backward with velocity v. The layers of air in contact with the ball spin with ball with velocity u. Therefore, the resultant velocity of air above the ball is $(v - u)$ and below the ball is $(v + u)$.



According to Bernoulli's theorem,

$$p + \frac{1}{2}\rho v^2 = \text{constant}$$

Therefore, pressure below the ball becomes lower than above the ball. Due to this pressure difference, a force acts on ball in downward direction. Therefore, the ball follows a curved path in spite of a parabolic trajectory.